Simplicity Versus Optimality

the choice of monetary policy rules when agents must learn

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Abstract

Economic theory tells us that advantages accrue to a central bank that commits to a policy rule. However, the rules that are discussed in the literature are not rules that are optimal in the sense of having been computed from an optimal control problem. Instead, the rules that are widely discussed--notably the Taylor rule--are remarkable for their simplicity. One reason for the apparent preference for simple *ad hoc* rules might be the assumption of full information that is generally maintained for the computation of an optimal rule. This tends to make optimal control rules less robust to model specification errors than are simple *ad hoc* rules. In this paper, we drop the full information assumption and investigate the choice of policy rules when private agents must learn the rule that is used. To do this, we conduct stochastic simulations on a small, estimated forward-looking model, with agents following a strategy of least-squares learning or discounted least-squares learning. We find that the costs of learning a new rule can, under some circumstances, be substantial. These circumstances vary with the preferences of the monetary authority and with the rule initially in place. Policymakers with strong preferences for inflation control must incur substantial costs when they change the rule in use; but they are nearly always willing to bear the costs of shifting to a (constrained) optimal rule. Policymakers with weak preferences for inflation control, on the other hand, may actually benefit from agents' prior belief that a strong rule is in place.

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1. Introduction

In recent years, there has been a renewed interest in the governance of monetary policy through the use of rules. This has come in part because of academic contributions including those of Hall and Mankiw (1994), McCallum (1987), Taylor (1993, 1994), and Henderson and McKibbin (1993). It has also arisen because of adoption in a number of countries of explicit inflation targets. New Zealand (1990), Canada (1991), the United Kingdom (1992), Sweden (1993) and Finland (1993) have all announced such regimes.

The academic papers noted above all focus on simple *ad hoc* rules. Typically, very simple specifications are written down and parameterized either with regard to the historical experience [Taylor (1993)], or through simulation experiments [Henderson and McKibbin (1993), McCallum (1987)]. Both the simplicity of these rules, and the evaluation criteria used to judge them stand in stark contrast to the earlier literature on optimal control. Optimal control theory wrings all the information possible out of the economic model, the stochastic shocks borne by the economy, and policymakers' preferences. This is, of course, a mixed blessing.

Optimal control theory has been criticized on three related grounds. First, the optimization is conditional on a large set of parameters, some of which are measured imperfectly and the knowledge of which is not shared by all agents. Some features of the model are known to change over time, often in imprecise ways. The most notable example of this is policymakers' preferences which can change either 'exogenously' through the appointment process, or 'endogenously' through the accumulation of experience.[1] Second, optimal control rules are invariably complex. The arguments to an optimal rule include all the state variables of the model. In working models used by central banks, state variables can number in the hundreds. The sheer complexity of such rules makes them difficult to follow, difficult to communicate to the public, and difficult to monitor. Third, in forward-looking models, it can be difficult to commit to a rule of any sort. Time inconsistency problems often arise. Complex rules are arguably more difficult to commit to, if for no other reason other than the benefits of commitment cannot be reaped if agents cannot distinguish commitment to a complex rule and mere discretion.

Simple rules are claimed to avoid most of these problems by enhancing accountability, and hence the returns to precommitment, and by avoiding rules that are optimal only in idiosyn-

1. Levin *et al.* (1998) examine the performance of rules in three models as a check on robustness of candidate rules.

cratic circumstances. At the same time, simple rules still allow feedback from state variables over time, thereby avoiding the straightjacket of 'open-loop' rules, such as Friedman's k-percent money growth rule. The costs of this simplicity include the foregone improvement in performance that a richer policy can add.

This paper examines the friction between simplicity and optimality in the design of monetary policy rules. With complete information, rational expectations, and full optimization, the correct answer to the question of the best rule is trite: optimal control is optimal. However, rational expectations can be expected to prevail only in the steady state, since only then will agents have sufficient knowledge to formulate a rational expectation. This means that changes in policy must consider not only the relative merits of the old and prospective new policies, but also the costs along the transition path to the new rule brought about by the induced break from rational expectations. With this in mind, we allow two elements of realism into the exercise that can alter the trite result. First, we consider optimal rules subject to a restriction on the number of parameters that can enter the policy rule--a simplicity restriction. We examine the marginal cost of this restriction. Second we restrict the information available to private agents, requiring them to learn the policy rule that is in force. In relaxing the purest form of the rational expectations assumptions, we follow the literature on learning in macroeconomics associated with Taylor (1975) and Cripps (1991) and advanced by Sargent (1993). We are then in a position to ask the question: if the Fed were to precommit to a rule in the presence of a skeptical public, what form should the rule take? If the Fed knew the true structure of the economy, would the rule that is optimal under full information still be optimal when private agents would have to learn the rule? Or would something simpler, and arguably easier to learn, be better in practice?

To examine these questions, we estimate a small forward-looking macro model with Keynesian features and model the process by which agents learn the features of the policy rule in use. The model is a form of a contracting model, in the spirit of Taylor (1980) and Calvo (1983), and is broadly similar to that of Fuhrer and Moore (1995b). We construct the state-space representation of this model and conduct stochastic simulations of a change in the policy rule, with agents learning the structural parameters of the linear monetary policy rule using recursive least squares, and discounted recursive least squares. Doing these sorts of experiments in an economy that is forward-looking obliges us to exploit modern efficient algorithms for computing state-space representations of the forward-looking model in real time.

The rest of this paper proceeds as follows. In Section 2 we discuss the simple, macroeconomic model. The third section outlines our methodological approach. Section 4 provides our results. The fifth and final section offers some concluding remarks.

2. The Model

We seek a model that is simple, estimated and realistic from the point of view of a monetary authority. Towards this objective, we construct a simple New Keynesian model along the lines of Fuhrer and Moore (1995b). The key to this model, as in any Keynesian model, is the price equation or Phillips curve. Our formulation is very much in the same style as the real wage contracting model of Buiter and Jewitt (1981) and Fuhrer and Moore (1995a). By having agents set nominal contracts with the goal of fixing relative real wages, the relative real wage formulation 'slips the derivative' in the price equation, thereby ruling out the possibility of costless disinflation.[2] However, instead of the fixed-term contract specification of Fuhrer-Moore, we adopt the stochastic contract duration formulation of Calvo. In doing this, we significantly reduce the state space of the model, thereby accelerating the numerical exercises that follow.

The complete model is as follows:

$$\pi_t = \delta\pi_{t-1} + (1-\delta)\, c_{t|t-1} \tag{1}$$

$$c_t = (1-\delta)\, [\pi_{t-1} + \gamma y_{t-1}] + \delta c_{t+1|t-1} + u_t^{\pi} \tag{2}$$

$$y = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 rr_{t-1} + u_t^{y} \tag{3}$$

$$rr_t = rs_t - \pi_{t|t-1} \tag{4}$$

$$R_t = rr^* + \pi_{t|t-1} + \beta_R\, [R_{t-1} - \pi_{t-1}] + \beta_\pi\, [\pi_{t-1} - \pi^*] + \beta_y y_{t-1} + u_t^{rs} \tag{5}$$

Equations (1) and (2) together comprise a forward-looking Phillips curve, with π and c measuring aggregate and core inflation respectively, y is the output gap, a measure of excess demand. The notation $m_{t|t-1}$ should be read as the expectation of variable m for date t, conditional on

2. Roberts (1995) shows that nearly all sticky price models can be boiled down to a specification where prices are predetermined, but inflation is not. By 'slipping the derivative' in the price equation, Taylor's specification which is two-sided in the price level becomes two-sided in inflation instead. We do not do disinflation experiments in this paper. Nevertheless, addressing policy questions of the sort considered here requires a reasonable characterization of the degree of inflation stickiness that is in the data.

information available at time *t-1*, which includes all variables subscripted *t-1*. Equation (1) gives inflation as a weighted average of inherited inflation, π_{t-1}, and expected core inflation, $c_{t|t-1}$. Following Calvo (1983), the expiration of contracts is given by an exponential distribution with hazard rate, δ. Assuming that terminations of contracts are independent of one another, the proportion of contracts negotiated s periods ago that are still in force today is $(1-\delta)\delta^{t-s}$. In equation (2), core inflation is seen to be a weighted average of future core inflation and a mark-up of excess demand over inherited inflation. Equations (1) and (2) differ from the standard Calvo model in two ways. First, as discussed above, the dependent variables are inflation rates rather than price levels. Second, the goods price inflation rate, π, and the output gap, y, appear with a lag (and leads) rather than just contemporaneously (along with leads). This specification more accurately captures the tendency for contracts to be indexed to past inflation, and for bargaining to take place over realized outcomes in addition to prospective future conditions. Equation (3) is a very simple aggregate demand equation with output being a function of two lags of output as well as the lagged *ex ante* real interest rate

Equation (4) is the Fisher equation. Finally, equation (5) is a generic interest rate reaction function, written here simply to complete the model. The monetary authority is assumed to manipulate the nominal federal funds rate, R, and implicitly deviations of the real rate from its equilibrium level, $rr - rr^*$, with the aim of moving inflation to its target level, π^*, reducing excess demand to zero, and penalizing movements in the instrument itself. Each of the state variables in the rule carries a weight of β_i, where $i = \{\pi, y, R\}$. These weights are related to, but should not be confused with, the weights of the monetary authority's loss function, about which we shall have more to say below.

The model is stylized, but it does capture what we would take to be the fundamental aspects of models that are useful for the analysis of monetary policy. We have already mentioned the stickiness of inflation in this model. Other integral features of the model include that policy acts on demand and prices with a lag. This rules out monetary policy that can instantaneously offset shocks as they occur. The model also assumes that excess demand affects inflation with a lag, and that disturbances to aggregate demand are persistent. These features imply that in order to be effective, monetary policy must look ahead, setting the federal funds rate today to achieve objectives in the future. However, the stochastic nature of the economy and the timing of expectations formation imply that these plans will not be achieved on a period-by-period basis. Rather, the contingent plan set out by the authority in any one period will have to be updated as new information is revealed regarding the shocks that have been borne by the economy.

We estimated the key equations of the model on U.S. data from 1972Q1 to 1996Q4.[3] Since the precise empirical estimates of the model are not fundamental to the issues examined here, we shall keep our discussion of them concise. One thing we should note is that we proxy $c_{t+1|t-1}$ with the median of the Michigan survey of expected future inflation. The survey has some good features as a proxy. It is an unbiased predictor of future inflation. At the same time, it is not efficient: other variables do help predict movements in future inflation. The survey also measures consumer price inflation expectations, precisely the rates that would theoretically go into wage bargaining decisions. The GDP price inflation used on the left-hand side of the equation can then be thought of as a pseudo-mark-up over these expected future costs. The disadvantage is that the survey is for inflation over the next twelve months, which does not match the quarterly frequency of our model.

3. Recursive least squares estimates indicated significant parameter instability prior to the early 1970s.

Table 1
Estimates of Basic Contract Model
(1972Q1 - 1996Q4)

description	label	estimate	t-statistic	summary statistics
$\pi = \left[1-(1-\delta)^2\right]^{-1}\left(\delta\pi_{t-1} + (1-\delta)^2\gamma u_{t-1} + (1-\delta)\,\delta c_{t,t+1}\right) + \Gamma' Z$ (A)				
Nixon price controls	$\hat{Z}_1$	**-5.35**	**(2.20)**	$R^2 = 0.97$
change in oil prices	$\hat{Z}_2$	**0.0019**	**(0.70)**	**SEE=1.02**
unemployment	$\hat{\gamma}$	**-0.23**	**(1.49)**	**B-G(1) = 0.01**
contract duration	$\hat{\delta}$	**0.41**	**(4.65)**	**Constrained linear IV**
$y = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 rr_{t-1}$ (B)				
first lag of output	$\hat{\phi}_1$	**1.25**	**(13.16)**	$R^2 = 0.88$ **SEE=1.21**
second lag of output	$\hat{\phi}_2$	**-0.36**	**(3.22)**	**B-G(1) = 0.04**
real fed funds rate	$\hat{\phi}_3$	**-0.14**	**(2.29)**	**OLS**
$u = \gamma_0 + \gamma_1 y_t + \gamma_2 T + \gamma_3 (poil_t/p_t)$ (C)				
output gap	$\hat{\gamma}_1$	**-0.32**	**(8.51)**	$R^2 = 0.80$ **SEE=0.60**
time trend	$\hat{\gamma}_2$	**0.0095**	**(1.94)**	**B-G(1) = 0.00**
relative price of oil	$\hat{\gamma}_3$	**0.86**	**(5.56)**	**2SLS**

Data: $\Delta\overline{poil}$change in oil prices is a four-quarter moving average of the price of oil imported into the U.S.; π is the quarterly change at annual rates of the chain-weight GDP price index; u is the demographically corrected unemployment rate, less the natural rate of unemployment from the FRB/US model database; c_{t+1} is proxied by the median of the Michigan survey of expected inflation, 12 months ahead; y is the output gap for the U.S. from the FRB/US model database; rr is the real interest rate defined as the quarterly average of the federal funds rate less a four-quarter moving average of the chain-weight GDP price index; $poil/p$ is the price of imported oil relative the GDP price index; and Nixon price controls equals unity in 1971Q4 and -0.6 in 1972Q1. All regressions also included an unreported constant term. Constants were never statistically significant. B-G(1) is the probability value of the Breusch-Godfrey test of first-order serial correlation.

Notes: Equation (A) is estimated with instruments: constant, time trend, lagged unemployment gap, four lags of the change in imported oil prices; two lags of inflation, lagged real interest rate, lagged Nixon wage-price control dummy, and the lagged relative price of imported oil. Standard errors for all three equations were corrected for autocorrelated residuals of unspecified form using the Newey-West (1987) method.

However, most of the predictive power of the survey to predict inflation over the next twelve months comes from its ability to predict inflation in the very short term rather than later on, suggesting that this problem is not serious. The estimates for three equations are presented in Table 1 above. Unemployment gaps--defined as the deviation of the demographically adjusted unemployment rate less the NAIRU--performed better than did output gaps, and so it appears in estimation of the Phillips curve. (The model is then supplemented with a simple Okun's Law relationship.)

For estimation purposes we embellished the basic formulation with a small number of exogenous supply shock terms, specifically oil prices, a variable to capture the effects of the Nixon wage-and-price controls, and a constant term. These are traditional and uncontroversial inclusions. For example, Roberts (1996) has found oil prices to be important for explaining the inflation in estimation using Michigan survey data.The key parameters are the 'contract duration' parameter, $\hat{\delta}$, and the excess demand parameter, $\hat{\gamma}$. If this were a *level* contracts model, $\hat{\delta} = 0.41$ would be a disappointingly low number since it implies a very short average contract length. This might be taking this interpretation too far, however. When equations (1) and (2) are solved, the reduced-form coefficient on lagged inflation is seen to be 0.846. This is substantial inflation stickiness by any measure and is similar to estimates of similar models by Laxton *et al.* (1998), Fuhrer (1997) and others.

3. Methodology

3.1 Optimal and Simple Policy Rules

It is useful to express the model in its first-order (state-space) form. To do this we begin by partitioning the vector of state variables into predetermined variables and 'jumpers', and expressing the structural model as follows:[4]

$$K_1 \begin{bmatrix} z_{t+1} \\ c_{t+1|t} \end{bmatrix} = K_0 \begin{bmatrix} z_t \\ c_t \end{bmatrix} + \begin{bmatrix} u_t^z \\ u_t^\pi \end{bmatrix} \tag{6}$$

where $z_t = \{R_t, \pi_t, y_t, y_{t-1}\}$ is the vector of predetermined variables, and c_t is contract inflation, our one non-predetermined variable. Constructing the state-space representation for the model for a given policy rule is a matter of finding a matrix, $K = K_1^{-1}K_0$ with properties that are

4. An able reference for state-space forms of linear rational expectations models and other issues in policy design is Holly and Hughes Hallett (1989)

desired by the monetary authority.

$$\begin{bmatrix} z_{t+1} \\ c_{t+1|t} \end{bmatrix} = K \begin{bmatrix} z_t \\ c_t \end{bmatrix} + K_1^{-1} \begin{bmatrix} u_t^z \\ u_t^\pi \end{bmatrix} \tag{7}$$

Equations (7) are recursive in the state variables so that manipulating the model is simple and computationally easy. However, two problems arise in the construction of K needed to do this.

The first of these problems is a technical one having to do with the fact that K_1 is often singular. We shall return to this later on. The second problem is more interesting from an economic point of view and concerns finding a specific rule with the desired properties. This is an exercise in optimal control. In the forward-looking context, however, the theory is a bit more complex than the standard textbook treatments. The optimal control rule is no longer a function just of the five state variables of the model as is the case with a backward-looking model. Rather, as Levine and Currie (1987) have shown, the optimal control rule is a function of the entire history of the predetermined state variables.[5] Even for simple models such as this one, the rule can rapidly become complex.

It may be unreasonable to expect agents to obtain the knowledge necessary to form a rational expectation of a rule that is optimal in this sense. Our experiments have shown that agents have great difficulty learning the parameters of rules with many conditioning variables. This is particularly so when some variables, such as contract inflation, c, and price inflation π, tend to move closely together. In small samples, agents simply cannot distinguish between the two; unstable solutions can often arise. Rather than demonstrate this intuitively obvious result, in this paper, we consider instead restrictions on the globally optimal rule, or what we call simple optimal rules.[6] Simple optimal rules are those that minimize a loss function subject to the model of

5. This is because the optimal control rule cannot be expressed as a function of the nonpredetermined variables since these 'jump' with the selection and operation of the rule. Rather, the rule must be chosen with regard to the variables that determine the jump. In the rational expectations context, this will be given by the entire history of predetermined state variables of the model. In many cases, this history can be represented by an error-correction mechanism, as Levine (1991) shows, but not always. In any case, even when an error-correction mechanism exists, the basic point remains that the complexity of the rule is significantly enhanced by the presence of forward-looking behavior.

6. The phrase 'simple rules' is borrowed from Levine (1991) who addresses issues similar to some of the ones considered here. We adopt it and add the word 'optimal' to signify that the parameterization of our rules is not *ad hoc*, but rather is determined from a well specified minimization problem as described below.

the economy--just as regular optimal control problems do--plus a constraint on the number of arguments in the reaction function.

For our purposes, we can state the monetary authority's problem as:

$$\underset{\langle\beta_\pi, \beta_y, \beta_R\rangle}{\text{argmin}} \; E_0 \sum_{i=0}^{\infty} \rho^i \left[\psi_\pi (\pi - \pi^*)^2 + \psi_y y^2 + \psi_{\Delta R} (\Delta R)^2\right] \tag{8}$$

subject to the state-space representation of the model as in equations (6), along with the consistent expectations restrictions: $m_{t+1|t} = m_{t+1}$ for all $m \subset \{c, z\}$, the arguments of the reaction function, (5): $R_t = rr^* + \pi_{t|t-1} + \beta_R [R_{t-1} - \pi_{t-1}] + \beta_\pi [\pi_{t-1} - \pi^*] + \beta_y y_{t-1} + u_t^R$.[7] The solution to this problem, which is described in some detail in the Appendix, is a vector of policy rule coefficients, $\bar{\beta} = \{\beta_y, \beta_\pi, \beta_R\}$ corresponding to the vector of objective function weights, $\bar{\Psi} = \{\psi_y, \psi_\pi, \psi_{\Delta R}\}$.

Rules that solve the above problem will, by definition, be inferior to the globally optimal rule that could have been derived using optimal control techniques, in the presence of full information. By the same reasoning, they will be superior to even simpler rules such as the Taylor rule, with coefficients that are not chosen according to optimization criteria. In our formulation, the Taylor rule is just our equation (5) with the added restrictions that $\beta_R = 0, \beta_y = \beta_\pi = 0.5$:[8]

$$R_t = rr^* + \pi_{t|t-1} + 0.5 [\pi_{t-1} - \pi^*] + 0.5 y_{t-1} + u_t^R \tag{9}$$

Equations (5) and (9) are the two policy rules that may be in force in our experiments. The coefficients in these rules are what agents are assumed to learn. The simple optimal rules derived from the above procedure are summarized in Table 2 below.

7. We also assume the existence of a commitment technology that permits the monetary authority to select and retain the rule that is optimal given the average set of initial conditions. Were we to assume that the monetary authority does not discount the future, we would not need a commitment technology. However, in order to conduct welfare comparisons along the transition path to different rules, we need to discount the future. The strength of our assumption regarding commitment technology will be directly related to the rate of discounting. We use a low discount factor of 0.9875, or 4 percent per year on a compounded annual basis.
8. We are taking a bit of license here, with our characterization of the Taylor rule, in that inflation in the original specification of the rule appears with a four-quarter moving average of inflation and both inflation and output appear contemporaneously. We take the former simplification to reduce the size of the state matrix for computational reasons.

3.2 Learning

Our generic policy rule, equation (5), represents a single row of the matrix in equations (7), the values of which evolve over time according to recursive linear learning rules. Let us re-express equation (5), slightly more compactly as:

$$R_t = \bar{\beta}_t x_t + \hat{u}_t^R \tag{10}$$

Equation (10) merely stacks the right-hand side arguments to the policy rule into a vector, $x_t = [y_{t-1}, \pi_{t-1}, R_{t-1}]'$, while allowing time variation in the *estimated* coefficients, $\hat{\beta}_t$. Let the time series of x_t be X_t; that is, $X_t = \{x_{t-i}|_0^t\}$ We assume that agents use either least squares or discounted least squares to update their estimates of $\hat{\beta}_t$. Harvey (1993, pp. 98-99) shows that the recursive updating formula in equation (11) is equivalent to continuous repetition of an OLS regression with the addition of one observation:

$$\hat{\beta}_t = \hat{\beta}_{t-1} + P_{t-1}x_t\left(R - x_t'\hat{\beta}_{t-1}\right)f_t^{-1} \tag{11}$$

where P_t is the 'precision matrix', $P_t = (X_t'X_t)^{-1}$, and $f_t = 1 + x_t'P_{t-1}x_t$ is a standardization factor used to rescale prediction errors, $\hat{u}_{rs,t} = P_{t-1}x_t\left(R - x_t'\hat{\beta}_{t-1}\right)$, in accordance with changes in the precision over time. The precision matrix can be shown to evolve according to:

$$P_t = \lambda P_{t-1} - P_{t-1}x_t x_t' P_{t-1} f_t^{-1} \tag{12}$$

The parameter λ is a 'forgetting factor'. The special case of $\lambda = 1$ discussed in Harvey (1993) is standard recursive least squares (RLS). If we assume that agents have 'less memory' we can downweight observations in the distant past relative to the most recent observations by allowing $0 < \lambda < 1$. This means that agents 'forget' the past at a geometric rate of $1 - \lambda$ percent per quarter. This is discounted recursive least squares (DRLS).

The memory of the learning system has a convenient Bayesian interpretation in that had there never been a regime shift in the past, agents would optimally place equal weight on all historical observations as with RSL. Under such circumstances, the Kalman gain in equation (11) goes to zero asymptotically, and $\hat{\beta} \to \bar{\beta}$. However, if there has been a history of occasional regime shifts, or if agents believe in such a possibility for other, external reasons, then the weight they discount previous estimates of the rule parameters can represent the strength of their prior belief. The lower is λ, the more likely agents believe regime shifts to be. If λ is taken as an exogenously fixed parameter (as we do throughout this paper) then $\hat{\beta}$ will have a tendency to fluctuate around $\bar{\beta}$, overreacting to surprises owing to the maintained belief that a regime shift has some likelihood of occurring. That agents overreact, in some sense, to "news" means agents with less memory will tend to learn new rules more rapidly, but this rapid learning is not necessarily welfare improving.

Whatever the precise form, using some form of learning here is a useful step forward since it models agents on the same plane as the econometrician, having to infer the law of motion of the economy from the surrounding environment, rather than knowing the law of motion *a priori*.

3.3 Numerical Issues

Choosing a policy rule to minimize a loss function, subject to a system of linear rational expectations equations, such as equations (6) plus the form of the rule give by equation (5), and the restriction that expectations are model consistent *from the point of view of private agents*, presents some computational difficulties. Under model consistent expectations, the solutions for current dated endogenous variables depend on expected future values of other endogenous variables, which are conditional on agents' (possibly incorrect) beliefs of what rule the monetary authority is using.

One such difficulty is that equations (7) must satisfy the Blanchard-Kahn (1980) conditions for the existence of a unique saddle-point solution to the system. The B-K conditions require that the number of eigenvalues greater than unity be equal to the number of non-predetermined variables in the system. Although there is just one non-predetermined variable in the model, the parameters of the model change with agent's perceptions of the rule that is being used. Instability or multiple equilibria are possibilities. There was some minor incidence of instability in our experiments, always associated with agents perceiving the coefficient on inflation less target inflation turning negative. Instability tended to arise more often when the short memory was employed in discounted recursive least squares learning, but never for full memory and rarely for reasonable amounts of memory in the learning process.[9] To keep the model stable, we restricted the perceived weight on inflation to be positive and included those trials in which this constraint was binding in our computations. However, tests on even the most egregious cases of instability

9. At levels of discounting of approximately $\lambda = 0.80$ and lower, instability became a significant problem. But discount factors this low imply a half life of memory of only about four quarters, which we would regard as implausibly low.

revealed very close to the same results whether or not those trials that breached the B-K conditions were included.

A second numerical issue that comes up, alluded to previously, concerns the problem of the possible singularity of the matrix K_1. Without a nonsingular K_1 matrix, the state-space representation of the model cannot be constructed, meaning that a recursive representation of the forward-looking model would not be possible. The model would then have to be solved using some extended path method such as the Fair-Taylor (1983) algorithm or its stacked-time counterpart, Laffargue (1990). Given that we are interested in stochastic simulations of the model with learning, the numerical cost of the using extended-path solution methods would be orders of magnitude more time consuming than a recursive method and would be subject to errors in measurement. Fortunately, the development in recent years of new methods of constructing state-space representations of linear forward-looking models obviates these complexities. For this paper, we employ the algorithm of Anderson and Moore (1985) which uses a QR decomposition to find a nonsingular equivalent to the matrix K_1.

Finally, the linearity of the model combined with the model consistency of expectations (from the point of view of private agents) ensures that the imposition of possibly incorrect terminal conditions is not an issue.

It is useful, at this point, to summarize our quantitative approach. Let us take the case where the monetary authority has been using the Taylor rule for some time and then shifts to a version of the simple optimal rule. The 'algorithm' for solving this problem can be summarized as follows:

(1) Initialize the variance-covariance matrix Ω associated with the residuals $\bar{u}$ at values based on historical estimates; set the initial date counter at $t=0$;

(2) Compute pre-experiment matrices P_0, X_0, $\bar{\beta}_0$ using the initial policy rule with private agents' expectations consistent with this rule;

(3) Substitute the coefficients for the pertinent simple optimal rule for the initial rule;

(4) For date t, solve the model for its state-space representation and check for satisfaction of the Blanchard-Kahn conditions (if the B-K conditions are not satisfied, set $\hat{\beta}_{\pi} = 0.01$);

(5) Draw random shocks for the stochastic residuals, $\bar{u}$, and set the nominal federal funds rate consistent with these shocks and the *true* policy rule, for date t;

(6) Simulate the model to find endogenous solution values, for t;

(7) Taking the solution values to step (6) as given, update agents' perceptions of the policy parameters;

(8) Repeat steps (4) though (8) for the next t, $t=\{1,2,...T\}$.

(9) When $t=T$, stop.

The next section discusses our results.

4. Simulation Results

One of our exercises is to consider the consequences of the *complexity* of rules for welfare, given that agents must learn the rule in use. The error-correction representation of the optimal control rule contains seven arguments. In principle, we could consider the difference in performance of this globally optimal rule with that of a very simple rule. However as we have already noted, it is difficult for agents to learn rules with large sets of variables, in small samples at least. Moreover it is difficult to convey results over a large number of parameters. To keep the problem manageable, we focus mostly on learning 3-parameter optimal rules, beginning from 2-parameter rules.

We also examine the costs of transitions from an *ad hoc* (sub-optimal) rule--our version of the Taylor rule--to an optimal 3-parameter rule, as well as from the optimal 2-parameter rule. This allows us to separate the costs of learning to be optimal, from learning complexity. Of course we could have chosen any of a number of rules that are suboptimal in the context of our model. Our choice reflects the familiarity of the Taylor rule to a large cross-section of economists involved in monetary policy debates. In addition, as Taylor (1993) argues, the Taylor rule approximates Fed behavior quite well over the 1980s.[10]

For our 3-parameter rules, we focus on the simple optimal rules described above, for a a variety of preferences, of which we concentrate on two different sets. All of the rules we consider can be considered inflation-targeting rules in that each places sufficient emphasis on a fixed target rate of inflation to ensure that this rate will be achieved, on average, over time. The two rules we concentrate on differ with regard to how aggressively they seek this goal, relative to other objectives. An authority that places a large weight on inflation stabilization--a weight of 0.8 in our formulation--is said to have strong preferences for inflation targeting. For conciseness, in most instances we shall call this authority "strong". Strong inflation-targeting authorities are weak output targeting authorities: They place a weight of 0.1 on output stabilization. Symmetrically, an authority that places a weight of 0.1 on inflation stabilization and a weight of 0.8 on output stabilization shall be referred to as having weak tastes for inflation targeting, relative to targeting output, and will be called "weak".[11]

With all sets of preferences, we place a weight of 0.1 on restricting the variability of the nominal federal funds rate. A monetary authority may be concerned with the variability of its instrument for any of a number of reasons. First, and most obviously, it may inherently prefer less variability as a matter of taste, either in its own right, or as a device to avoid criticism of excessive activism. Second, constraining the volatility of the federal funds rate might be a hedge against model misspecifications, both fundamentally, in the sense of missing variables and the like, or more broadly owing to reluctance to using point elasticities estimated over a narrow range of

10. Our quantitative work has shown the Taylor rule both as it is written in Taylor (1993), and as we have written it here, works satisfactorily in a wide range of models.

11. Nothing pejorative should be taken from our calling authorities that place a large weight on inflation stabilization as "strong" or calling authorities that place a large weight on output stabilization as "weak" since these labels apply only to their *relative* tastes for inflation stabilization. There is complete symmetry between preferences for output versus inflation stabilization. This means that we could have called the strong inflation targeter a weak output targeter. Both policies yield the same average inflation rate.

movement of the federal funds rate being applied over a much wider range of contemplated variation. These taste parameters and the rule coefficients they imply are summarized in Table 2.

Finally, we conduct many of these experiments using three different degrees of memory. The first is full recursive least squares, $(\lambda = 1)$, which we shall refer to as 'full memory'. The other two cases are discounted recursive least squares (DRLS), one with somewhat limited memory, $(\lambda = 0.95)$ which we will call 'long memory' and another with 'short memory': $(\lambda = 0.85)$. The long memory case corresponds with a mean lag of 19 quarters, a reasonable length of time for a regime. The short memory case has a mean lag of only about six quarters and represents a public that is very skeptical of the constancy of any policy regime. In this way, these three choices bracket the possible linear learning strategies quite well.

4.1 The rules in the steady state:

Let us build up some intuition for what follows with an examination of the optimal policy rules in steady state. The parameters of the strong and weak inflation-targeting monetary authority's rules are summarized in Table 2 below, along with the parameters of our version of the Taylor rule. The mapping of penalties, measured by the $\psi_i, i = \{y, \pi, \Delta R\}$, and the *relative* magnitudes of the policy rule parameters, $\beta_j, j = \{y, \pi, R\}$, is as one would expect: A larger the coefficient on, say, ψ_π, results in a larger coefficient on β_π relative to β_y.

The performance of the complete set of rules with $\psi_{\Delta R} = 0.1$ and preferences for the range of weights on ψ_y, ψ_π bounded between 0.1 and 0.8, with $\psi_y + \psi_\pi = 1$, are summarized by Figure 1. The figure shows the trade-offs between the standard deviations of inflation and output in the steady state in the right-hand panel, and between inflation variability and the variability of the change in the federal funds rate in the left-hand panel.

First make note of the lower three curves in the right-hand panel. The solid line is the frontier for the globally optimal rule--the one in which seven state variables appear. The dot-dashed frontier immediately to the north east of the globally optimal frontier is associated with optimal 3-parameter rule, while the dashed frontier is the optimal 2-parameter frontier. The strong and weak rules are, respectively, at the bottom and top of each of the curves, with SO representing the strong rule generated from optimal control, S2 the optimal 2-parameter strong rule, and so on.[12]

12. Notwithstanding that the strong and weak rules appear at the end of the curves shown in the figure, these are the most extreme rules that are calculable. We have limited the range of the curves to keep the scale of the figures reasonable.

The first and most obvious thing to note about these frontiers is that the globally optimal rule is closest to the origin, meaning it produces superior performance to all other rules, measured in terms of output and inflation variability. A more interesting result is that the 3-parameter and 2-parameter frontiers cross. This crossing reflects the fact that movement along the 2-parameter rule frontier towards more strong preferences involves a much larger sacrifice in terms of higher interest-rate variability than for the 3-parameter rule. Evidently, inflation control and interest-rate control are strong substitutes, at the margin, for the strong policymaker. That this is so in the neighborhood of the strong rules, and not so in the vicinity of the weak rules, is an observation we shall return to below when we discuss transitions from 2- to 3-parameter rules in subsection 4.3.

The point marked with a 'T' in right-hand panel is the Taylor point representing the performance of the Taylor rule in this model. This rule performs substantially worse than any optimal 2-parameter rule in terms of output and inflation variability, but at 3.0 percentage points, produces a substantially lower standard deviation of the change in the federal funds rate.[13] It follows that there is a set of preferences for which the Taylor rule is optimal in this model. Some computation reveals that a coefficient of about 0.5 on the change in the federal funds rate produces an optimal frontier that slices through the Taylor point as shown. Thus, a policy maker that uses the Taylor rule is very reticent to allow fluctuations in the policy instrument.

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13. This is not shown in the left-hand panel of the figure in order to maintain visual clarity.

Table 2
Coefficients of Simple Optimal Rules

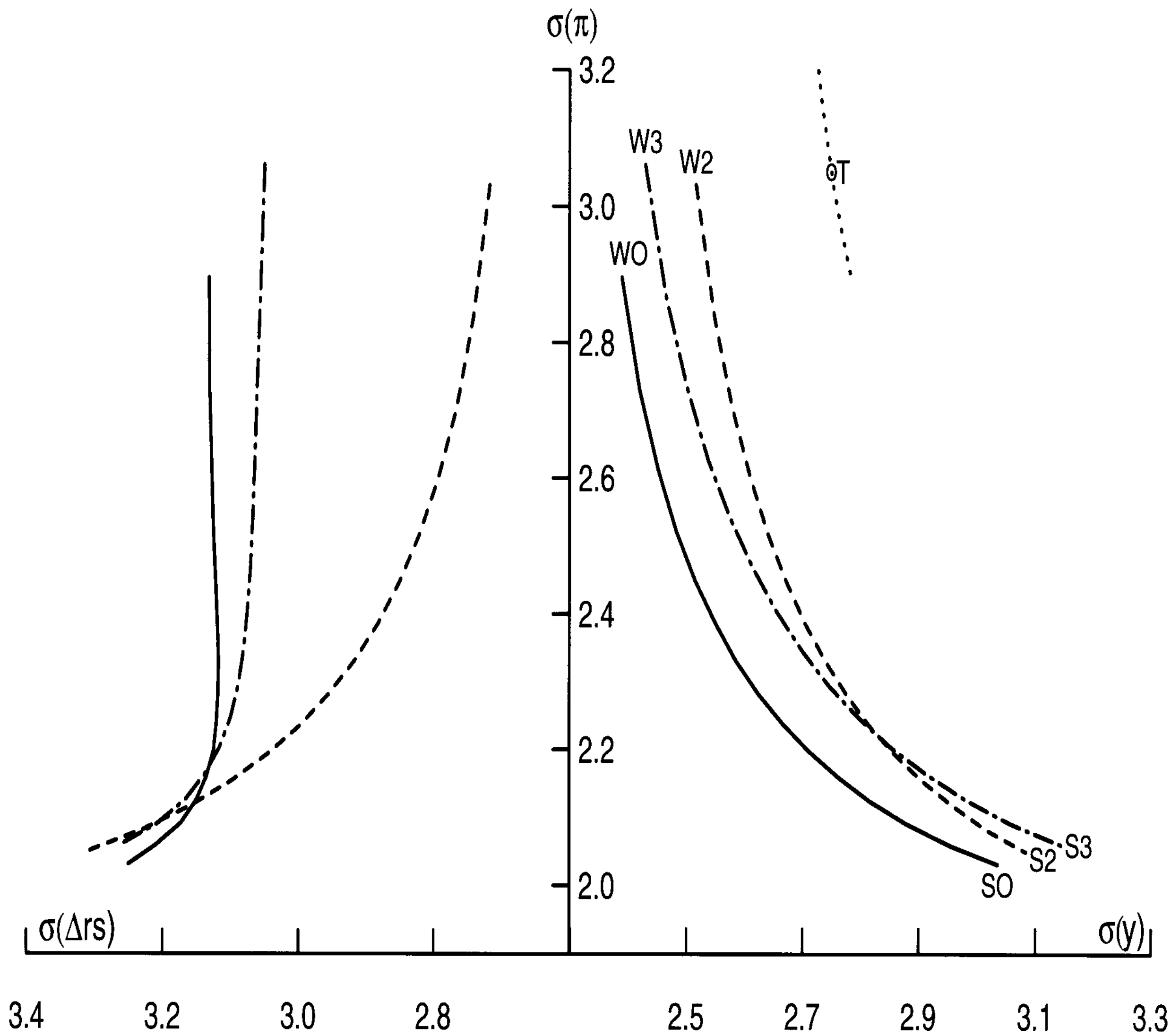
Preferences		Policy Rule Parameters		
$\{\psi_y, \psi_\pi, \psi_{\Delta R}\}$	**no. of arguments**	β_y	β_π	β_R
Taylor rule	**2 parameter**	**0.5**	**0.5**	**n/a**
strong $\{0.1, 0.8, 0.1\}$	**2 parameter**	**0.73**	**1.70**	**n/a**
	3 parameter	**0.60**	**1.35**	**0.27**
weak $\{0.8, 0.1, 0.1\}$	**2 parameter**	**1.18**	**0.41**	**n/a**
	3 parameter	**1.55**	**0.53**	**-0.33**

Notes: rule parameters are the solutions to the problem of minimizing equation (8) subject to (6) and (5) under model consistent expectations, for the loss function weights shown in the first column of the table. The terms 'strong' and 'weak' refer to the strength of preferences for inflation stabilization, relative to output stabilization.

The steepness of the frontier at the Taylor point also shows that this version of the Taylor rule is very weak in that a monetary authority that would choose this point is apparently willing, at the margin, to accept a very large increase in inflation variability to reduce output variability only slightly. While this result is only literally true for our characterization of the Taylor rule, and only within this model, the same general conclusions can be drawn from a similar exercise using the original form of the rule from Taylor (1993) within larger, more complex models.[14]

14. For example, the same general finding that the Taylor rule is 'weak' in the sense described in the text and averse to interest-rate variability arises with the FRB/US model of the U.S. economy maintained by the Federal Reserve Board, albeit somewhat less dramatically so. This finding may reflect either the policy preferences in play over the period to which the Taylor rule was loosely fitted, or it may reflect the relatively placid nature of the shocks over the period.

Figure 1
Selected Optimal Policy Frontiers



4.2 Changes in Preferences:

Having examined the steady-state performance of our rules, let us now consider transitions from one rule to another. In order to keep things simple, we shall focus initially on changes in policy preferences using optimal 2-parameter rules. In particular, the thought experiment we have in mind is of a newly installed policymaker, replacing an incumbent of quite different preferences. The new authority has to decide whether to switch to a new rule that is consistent with its preferences. The new rule will produce better steady-state performance--from the point of view of the new authority--but presumably only after bearing some transitional costs as agents learn of the new rule. It is conceivable that the transitional costs could be so high that the new authority would prefer to continue to govern policy with the old rule, even though it is inconsistent with its steady-state preferences.[15]

Figure 2 shows the evolution over time of the parameter estimates for the transition from a regime of weak preferences for inflation control, to a strong one, using the 2-parameter optimal rules. The upper panel shows the perceived coefficient on excess demand and the lower panel shows the coefficient on inflation. In each panel, we also show a base case--the solid line--where agents believe the initial regime will continue, and they are correct in this expectation. For all the other cases, their prior expectations turn out to be incorrect. The dashed line is the 'full memory' case. The dotted line is 'long memory' learning case (that is, discounted recursive least squares learning with $\lambda = 0.95$). The dot-dashed line is 'short memory' case (DRLS with $\lambda = 0.85$). The lines in each case are the average values over 4,000 draws. Each simulation lasts 200 periods.[16]

Let us concentrate, for the moment, on the upper panel of Figure 2. The first thing to note from the figure is the obvious point that agents do not get fooled in the base case. That is, if the rule is the 2-parameter optimal weak rule, purely random shocks do not induce agents to erroneously revise their estimates of the perceived rule, when agents learn without discounting. This is not a trivial result since, as we shall see, it is not as clear when agents 'forget' past observations.

15. This experiment is broadly similar to Fuhrer and Hooker (1993) who also consider learning policy parameters. However, they do not consider optimal rules and do not examine the welfare implications of the choice of rules.

16. In order to ensure the lags and the precision matrix were properly initiated according to pre-experiment conditions, 200 periods were simulated using the initial rule before shifting to the new rule. For each experiment, 300 periods were simulated and each experiment comprised 3000 draws so that altogether each of the major experiments reported in this paper involved computing 900,000 points. On a single processor of an UltraSparc 4 296 megahertz UNIX machine, each experiment took a bit over 3 hours to compute.

Figure 2
Perceived Policy Rule Parameters
Learning a Shift from 'Weak' to 'Strong' Inflation-Control Preferences
(average of 4000 draws)

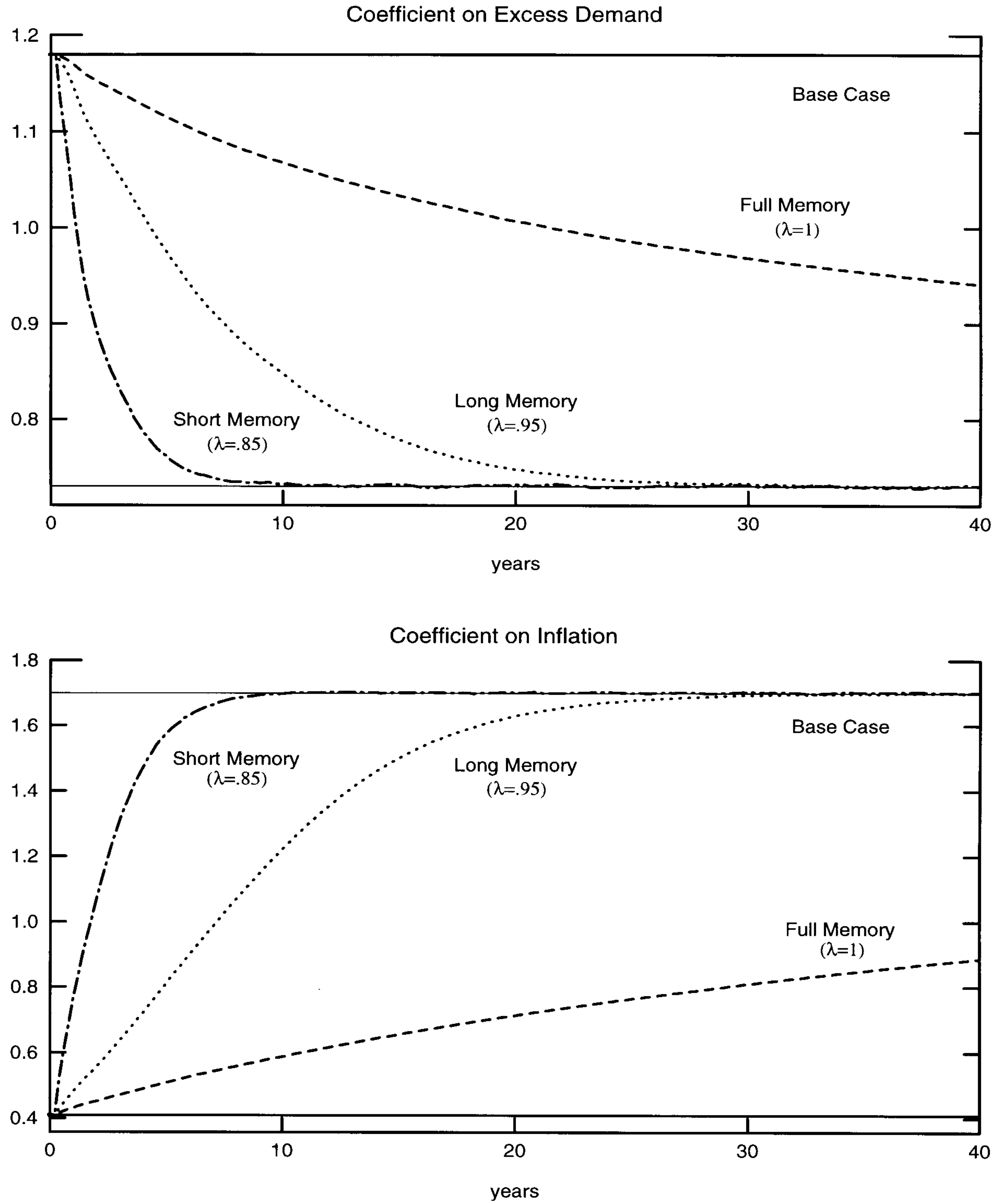
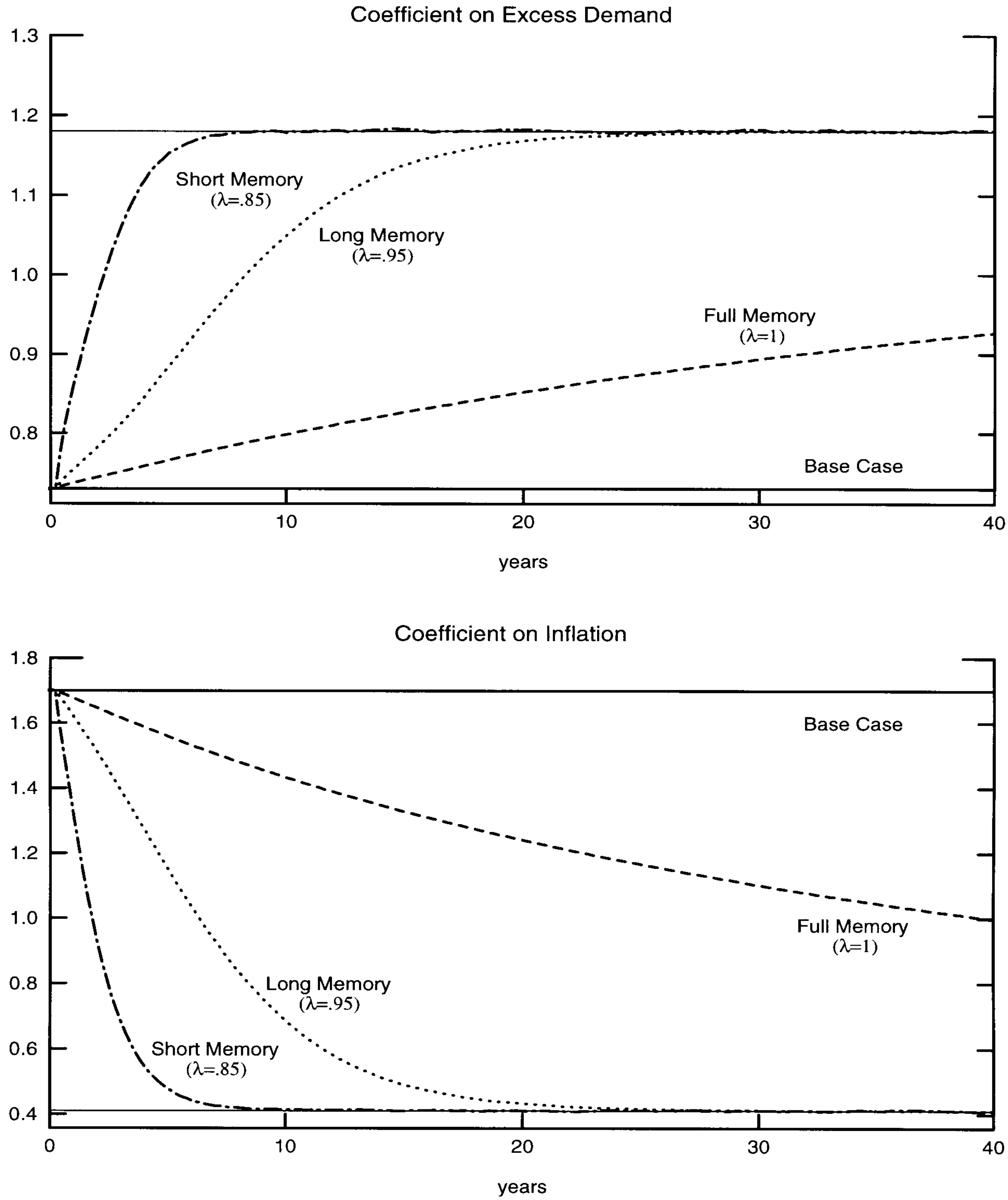


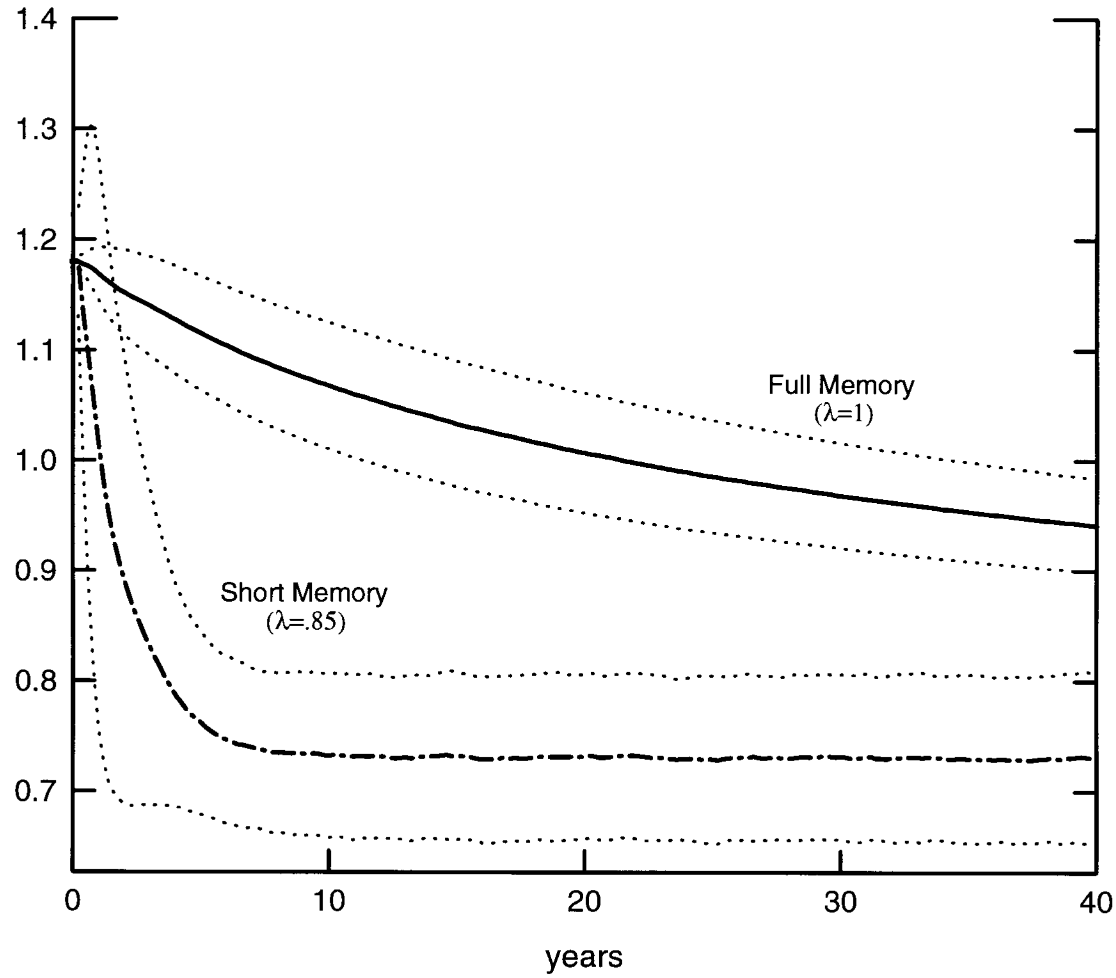
Figure 3
Perceived Policy Rule Parameters
Learning a Shift from 'Strong' to 'Weak' Inflation-Control Preferences
(average of 4000 draws)



More important than this, however, is the observation that regardless of which of the three learning devices that is used, it takes a remarkably long time for agents to come to grips with the change in parameters. In particular, with full memory, agents have not learned the true rule coefficients even after the fifty years covered in the experiment. Even under rapid discounting of $\lambda = 0.85$ --meaning that the mean lag of the memory process is just 5.7 quarters--it takes more than ten years before agents reach the new parameter values. This finding, which is consistent with earlier results, such as those of Fuhrer and Hooker (1993), stems from two aspects of the learning rules. First, the speed of updating is a function of the signal-to-noise ratio in the learning mechanism. Because a considerable portion of economic variability historically has come from random sources, agents rationally infer the largest portion of surprises to the observed federal funds rate settings as being noise. Accordingly, to a large extent, they ignore the shock. Second, these results show how linear learning rules tolerate systematic errors; the forecast errors that agents make get increasingly smaller, but are nonetheless of the same sign for extended periods of time. A non-linear rule might react not just to the size of surprises, but also a string of errors of one sign.

The bottom panel of Figure 2 shows the evolution of the perceived coefficient on inflation. Figure 3 shows the analogous learning rates when a strong inflation-targeting regime is succeeded by a weak one. Very much the same conclusions can be drawn from these figures as is drawn from the upper panel of Figure 2.

Figure 4
Perceived Coefficient on Excess Demand
Learning a Shift from 'Weak' to 'Strong' Inflation-Control Preferences
(with +/- 1 standard error bands)



We can derive a third observation, this time from Figure 4, which repeats some of the detail of Figure 2, but adds confidence intervals of plus-or-minus one standard error around the coefficient means for the full-memory and short-memory cases. The bands show us that as agents increasingly discount the past in forming their expectations, the coefficient estimates become less and less precise, particularly in the early going of the learning exercise. Also, as one might expect, there is considerably more variability around the steady-state estimates of the policy parameters under the short memory case than there is under the full memory case. The figure does not show the full memory case after it converges on the true policy parameters but it is evident from the figure, and demonstrable from the data, that the variability of $\hat{\beta}$ is lower in the steady state under full memory than it is under short memory.

Table 3 summarizes the welfare implications of this experiment. Since other tables that follow this one are broadly similar in construction to Table 3, there are dividends to be reaped from taking some time to explain in detail how to read this table. The table is divided into two panels. The top panel shows the decision a strong inflation-targeting policy maker would have to make after succeeding a policy maker with weak preferences for inflation targeting. The row in this panel labeled 'base case' shows the performance that the economy would enjoy had the strong policy rule been in effect at the outset. This can be thought of as the 'after picture'. The second row, labeled 'weak' shows the performance of the economy under the weak rule; that is, the rule that was in place when the strong policymaker took over. This is the 'before picture'. Finally, the third row showing the same for the transition case from the weak rule to the strong rule. The key column of the panel is the far right-hand column showing welfare loss. This is the average discounted loss under the policies, *measured from the perspective of the incoming strong policymaker.*[17] In terms of the raw numbers, the performance of the economy under, say, the weak rule as shown in the second row, will not be any different than it was under the weak regime, but the loss ascribed to this performance can differ markedly depending on the perspective of the policymaker..

17. Loss figures shown are the average over the 4,000 draws conducted. A quarterly discount factor of 0.9875, or about six percent a year, is applied in computing the losses. This is a modest discount factor, in line with what might be used in financial markets. The substantive facts presented in this paper were invariant to the choice of discount factors, at least within a range we would consider to be reasonable. Political economy arguments might yield positive arguments for a substantially lower discount factor, but this is not the subject of this paper.

Table 3
Simulation Results from Change-in-Preference Learning Exercises
(Average across 4000 draws)

Rule in use	Standard Deviation of: π	y	R	$\rho(\pi, y)$	Autocorrelation of: π	y	R	Welfare Loss L
'strong' inflation-targeting preferences, full memory, 2-parameter rules								
base case	**2.1**	**3.0**	**6.4**	**.31**	**.75**	**.84**	**.83**	**1**
weak rule	**2.9**	**2.5**	**5.6**	**.21**	**.86**	**.81**	**.86**	**1.56**
lib --> con	**2.3**	**3.2**	**7.0**	**.22**	**.78**	**.85**	**.84**	**1.28**
'weak' inflation-targeting preferences, full memory, 2-parameter rules								
base case	**2.9**	**2.5**	**5.6**	**.21**	**.86**	**.81**	**.86**	**1**
con rule	**2.1**	**3.0**	**6.4**	**.31**	**.75**	**.84**	**.83**	**1.31**
con --> lib	**2.4**	**2.5**	**5.1**	**.33**	**.81**	**.80**	**.84**	**0.93**

Notes: The first row of each panel contains the 'base case' results defined as those corresponding to the optimal 2-parameter rule for the policy preferences noted; the row immediately below each base case is the performance of, and loss to, staying with the 2-parameter policy rule inherited from the previous regime. The third row shows the performance and cost of changing from the inherited regime to the (new) optimal 2-parameter rule. Qualitatively similar results were derived using 3-parameter rules and using discounted recursive least squares learning.

To aid comparison, we have normalized the loss figures for the base case to unity. By comparing the third row of the upper panel with the first row, we can see the loss associated with having to learn the new rule. By comparing the third row with the second row, we can see whether the transition costs of learning the new rule are so high as to induce the new policymaker to stay with the old rule. The rest of the columns of the table report some summary statistics that are useful in interpreting the results. With the exception of the center column, these are largely self explanatory. The center column shows the statistical correlation of output and inflation in the simulated data. Comparing this across experiments gives an indication of the extent to which the monetary authority is using the Phillips curve trade-off to achieve its objectives.

The bottom panel of the table is analogous to the top panel, except that the transition is from a former strong inflation-targeting regime to a new weak regime.

Turning to the results themselves, in the case of the incoming strong policymaker, the process of learning is costly relative to the base case; not being able to simply announce a new policy regime and have private agents believe it, implies substantial costs. However, the comparison of rows two and three shows that the incoming strong policymaker would be willing to bear the transition costs of switching rather than stick with the incumbent rule. Given the stark differences in preferences that these two policymakers represent, this is not particularly surprising. It is easy to show, however that at least some moderate policymakers--those that place a larger weight on output stabilization than a strong policymaker would, but not as large a weight as the weak policymaker does--would also not want to bear the transition costs of adjusting to a new, moderate rule, after succeeding a weak policymaker.

A more intriguing case is the transition from a strong policy regime to a weak one, shown in the lower panel. The third row of the lower panel shows that the weak policymaker actually benefits from the prior belief of private agents that a strong rule is in place: the loss under the transition, at 0.93, is lower than the normalized loss of unity in the base case. On the surface, this is surprising since, by definition, the base case should yield the best result possible in the circumstances. The reason why this intuition does not hold becomes clear once one recognizes that learning is breaking, in some sense, the model-consistency condition in the model. Notice that lower loss in the transition case comes from reduced inflation and federal funds rate volatility.[18]

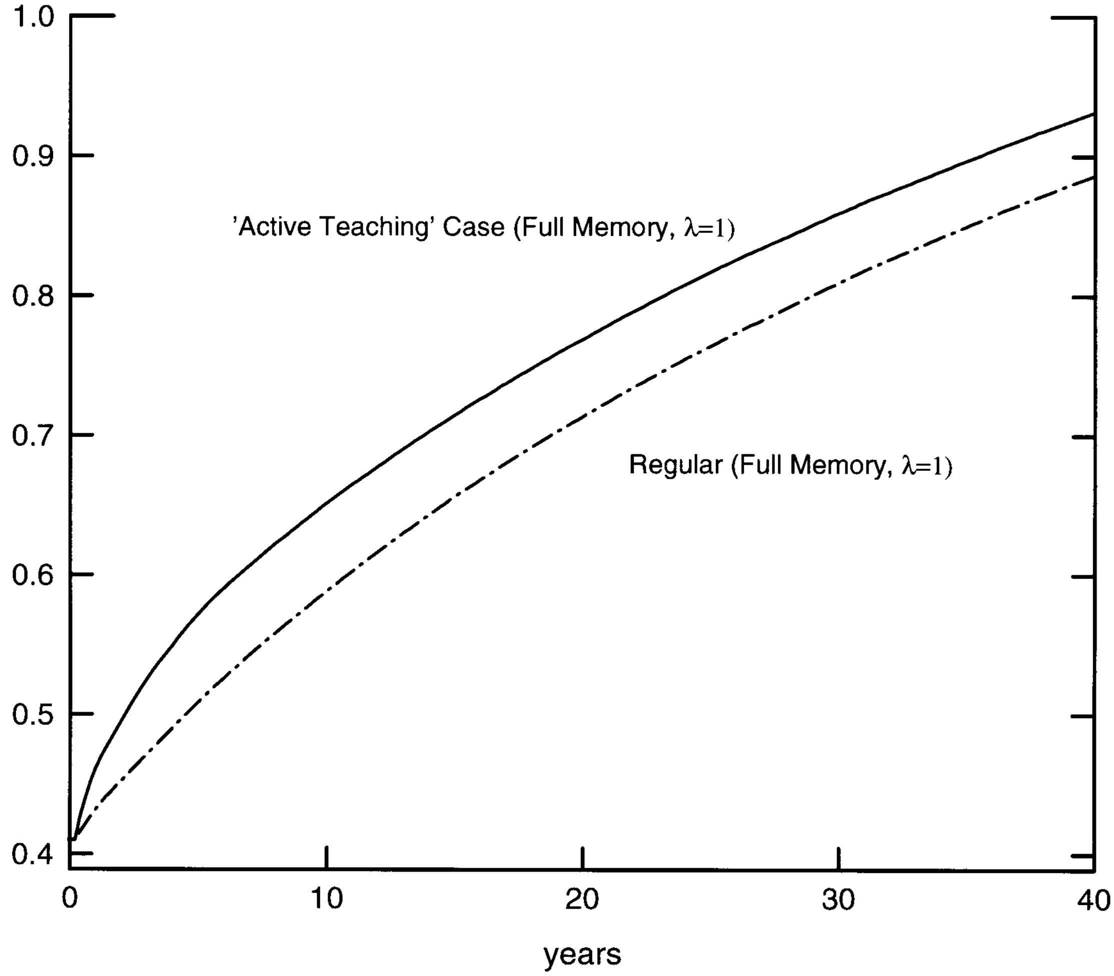
The volatility of inflation is, to a large extent, determined by the expected future path of the model's jump variable, contract inflation, $c_{t,t+1}$. However, future contract inflation is being pinned down by the expectation that monetary policy in the future will contain future inflation with the optimal strong rule. That this expectation turns out to be erroneous is of no consequence. From the policymaker's point of view, expectations are not model consistent, even though they are from the point of the view of private agents. Thus, to the extent that learning is slow, the weak policymaker can indulge his inclination to manage output without bearing the costs of incipient inflation pressures, and with reduced interest rate variability as well. No similar benefit is shown for the transition from weak to strong preferences shown in the upper panel of the table. Two rea-

18. The table actually shows the standard deviation of the level of the federal funds rate while the loss function contains the variance of the change in the federal funds rate, but the qualitative comparison is still valid.

sons account for this asymmetry. The first of these relates to the fact that the monetary authority controls inflation primarily through its management of aggregate demand. To a substantial degree, the belief by private agents that the authority will manage output tightly is only beneficial to the strong authority to the extent that inflation fluctuations originate from demand shocks. A large portion of inflation variability in the U.S. economy comes from shocks to the Phillips curve--that is, from so-called supply shocks. The larger the proportion of shocks to inflation originating from supply-side sources, the more conflicts arise in the management of demand and the control of inflation. This effect is not at work when moving from the strong to weak preferences, because demand management comes at an earlier stage in the monetary transmission mechanism than does inflation control.The second reason is that there is no jump variable in output in this model. This leaves expected future demand less important for current demand management than is the case for inflation. [19] It follows that a strong monetary authority wants to credibly announce regime changes when taking over from a weak incumbent, while a newly installed weak policymaker wants to conceal his true preferences and possibly forestall private agents' learning as much as possible. The observation that weak policymakers gain from being perceived as strong, because doing so decouples inflation expectations from policy actions for a time, echoes the theoretical literature on policy games. In that literature, policymakers do not find it beneficial to reveal their true preferences, as in Cukierman and Meltzer (1986) and Vickers (1986) for example. Similarly, in the "cheap talk" literature exemplified by Stein (1989), the central bank finds that sending vague signals tends to dominate full disclosure as a policy.

19. . We believe, but cannot prove at this point, that this second factor is less important than the first. The candidate jump variable in output is a permanent shock to the level of total factor productivity (or some similar supply shock) which, to a first approximation, should shift both actual and potential output by similar amounts, leaving the output gap more or less unchanged. If the goal of the weak authority is to control excess demand, this should be of second-order importance.

Figure 5
Perceived Coefficient on Inflation
Learning a Shift from 'Weak' to 'Strong' Inflation-Control Preferences
(average of 4000 draws)



Before we leave this subsection, let us consider the possibility that the monetary authority might be able to aid its cause by taking action to speed learning along the transition from one policy regime to another. In particular, we assume that the authority engages in what we shall call 'active teaching': aggressively signalling the change in policy by accentuating differences in policy. To do this operationally, we assumed that the newly installed authority chooses interest-rate surprises that are initially three times larger than in the regular experiment depicted in Table 3. We then allow this exaggerated policy signalling to slowly decline to normal over time.[20] Figure 5 shows the results for the speed of learning, in this case for the perceived inflation parameter, following a shift from a weak 2-parameter rule to a strong rule. As one might expect, the monetary authority can induce faster learning by private agents.

But can the authority increase welfare? Table 4 below summarizes the welfare implications of this experiment. The table shows a quantitatively important deterioration in performance relative to the regular case transition. The reasons for this are two-fold. First, in order to initiate the higher rates of learning, the monetary authority must inject federal funds rate volatility into the economy. This counts directly as a negative in the loss function. On top of this, however, the extra shocks do improve the performance in the variable that the new regime cares about most--inflation for the strong authority and output for the weak inflation-targeting authority--but not forever. Moreover, the extra shocks are disastrous for the variable that the new authority cares about least. Only with heavy discounting of the future would such a strategy of active learning pay off--at least in the blunt form that we have introduced it here.

20. Specifically, think of a 'surprise function' $s(\hat{\beta}-\bar{\beta})$ where surprises vary directly with the discrepancy between the actual and perceived rule parameters. The 'active teaching' scenario replaces this surprise function with $s(\hat{\beta}-\bar{\beta}) + \alpha(s(\hat{\beta}-\bar{\beta}))\theta^t$ where $\alpha = 2$, $\theta = 0.925$ and $t = 0, 1, 2, \ldots$

Table 4
Implications of 'Active Teaching' for Change-in-Preference Learning
(Average across 2000 draws)

experiment	Welfare Loss L	experiment	Welfare Loss L
full memory, 2-parameter rules			
conservative --> liberal		**liberal --> conservative**	
base case	**1**	**base case**	**1**
regular experiment	**0.93**	**regular experiment**	**1.28**
'active teaching'	**0.98**	**'active teaching'**	**1.39**

Notes: 'regular experiment' is the transition from a conservative 2-parameter optimal rule to a liberal 2-parameter optimal rule (two left-hand columns) or vice versa (two right-hand columns). 'active teaching' is where the surprises along the rule transition have been accentuated by three-fold initially with this accentuation declining at a quarterly geometric rate of 0.925 thereafter. Losses are discounted losses with a discount factor of 0.9875 and have been normalized around the base-case loss.

It is conceivable that a fully optimizing authority could choose the rate of learning directly over time and in so doing produce a better performance than the regular case transition shown here. In this regard, Cripps (1991) shows that an optimizing central bank might want to *slow* down the rate of learning private agents. The case shown in the bottom part of Table 3 immediately suggests some likelihood of that in the current set-up, at least as a special case.

4.3 Learning to be Optimal:

Now let us consider agents who begin with a 2-parameter rule--in the present example, the Taylor rule--who must then learn that the Fed has shifted to one of our simple optimal rules laid out in Table 2 above. We consider first the results for strong inflation-targeting preferences, summarized in Table 5 below. For ease of comparison we shade the base-case row, which in this case is the 3-parameter optimal rule, and normalize the loss to unity.[21] The first panel of the table shows the performance of the economy under the three alternative steady states; it corresponds to Figure 1 above. Having already examined Figure 1, it is not surprising that the Taylor rule's per-

21. The base-case loss is independent of the 'memory' in the learning process.

formance features substantially higher inflation variability than the strong base-case rule, but lower variability in output and especially the federal funds rate. The Taylor rule's performance also shows substantially more persistence in inflation and interest rates (not shown) and a higher correlation of output and inflation indicating that policy is working more through the traditional Keynesian channel, rather than through expectations, than in the base case. From the point of view of a strong policymaker, the Taylor rule is seen as a poor performer. Measured in terms of discounted loss, the Taylor rule is 56 percent worse than the 3-parameter optimal rule. Putting the same performance a different way, the strong policymaker would just as soon accept an autonomous increase in the variance of inflation of three-quarters of a percentage point or a whopping 3.1 percent in output variability as be forced to use the Taylor rule.[22]

The second row of the table shows the performance of the optimal 2-parameter rule. Relative to the base case, there is only a two-percent loss from using the optimal 2-parameter rule. The strong authority would be willing to sacrifice output variability of 0.2 percent in order to avoid using this rule. While this is not trivial, it is also not large. Thus, a significant finding in this paper is that the gains in steady state from using even slightly more complex rules than a 2-parameter rule is small--at least for strong preferences--provided that the simpler rule's parameters are chosen optimally.[23] The optimal 2-parameter rule has the same arguments as the Taylor rule, but markedly different coefficients, as Table 2 shows. Obviously there are large gains to be had from picking rule parameters judiciously.

22. This and all subsequent equivalent variation calculations are measured relative to the base-case rule.
23. Tetlow and von zur Muehlen (1996) report a similar finding with another small model as does Williams (1997) with the FRB/US model which has some 300 equations.

Table 5
Learning under Strong Preferences for Inflation Targeting
(Average across 4000 draws simulated for 200 periods each)

Rule(s)	Standard Deviation: π	y	R	$\rho\ (\pi, y)$	Loss L
base case results					
Taylor rule	**3.0**	**2.7**	**5.1**	**.39**	**1.58**
optimal 2 parameter	**2.1**	**3.0**	**6.4**	**.31**	**1.02**
optimal 3 parameter	**2.0**	**3.1**	**6.6**	**.35**	**1**
learning with full memory ($\lambda = 1$)					
Taylor --> 3 parameter	**2.4**	**3.4**	**7.6**	**.35**	**1.37**
Taylor --> 2 parameter	**2.3**	**3.1**	**7.1**	**.31**	**1.29**
2 --> 3 parameter	**2.1**	**3.2**	**6.7**	**.35**	**1.04**
learning with long memory ($\lambda = 0.95$)					
Taylor --> 3 parameter	**2.2**	**3.2**	**6.8**	**.35**	**1.17**
Taylor --> 2 parameter	**2.1**	**3.0**	**6.6**	**.31**	**1.15**
2 --> 3 parameter	**2.1**	**3.1**	**6.6**	**.35**	**1.02**
learning with short memory ($\lambda = 0.85$)					
Taylor --> 3 parameter	**2.1**	**3.2**	**6.6**	**.35**	**1.07**
Taylor --> 2 parameter	**2.1**	**3.0**	**6.4**	**.31**	**1.07**
2 --> 3 parameter	**2.1**	**3.1**	**6.6**	**.35**	**1.01**

Notes: The syntax "n--> m" refers to the results from learning the transition from the n-parameter simple optimal rule to the m-parameter simple optimal rule. Losses are computed as discounted loss with a discount factor equal to 0.9875.

Table 6
Learning under Weak Preferences for Inflation Targeting
(Average across 4000 draws simulated for 200 periods each)

Rule(s)	Standard Deviation: π	Standard Deviation: y	Standard Deviation: R	$\rho\ (\pi, y)$	Loss L
base case results					
Taylor rule	**3.0**	**2.7**	**5.1**	**.39**	**1.12**
optimal 2 parameter	**2.9**	**2.5**	**5.6**	**.21**	**1.02**
optimal 3 parameter	**3.0**	**2.4**	**5.6**	**.20**	**1**
learning with full memory ($\lambda = 1$)					
Taylor --> 3 parameter	**2.8**	**2.4**	**5.6**	**.27**	**0.99**
Taylor --> 2 parameter	**2.8**	**2.5**	**5.6**	**.29**	**1.01**
2 --> 3 parameter	**2.9**	**2.4**	**5.5**	**.21**	**0.99**
learning with long memory ($\lambda = 0.95$)					
Taylor --> 3 parameter	**2.9**	**2.4**	**5.6**	**.22**	**1.00**
Taylor --> 2 parameter	**2.9**	**2.5**	**5.6**	**.23**	**1.01**
2 --> 3 parameter	**2.9**	**2.4**	**5.6**	**.20**	**1.00**
learning with short memory ($\lambda = 0.85$)					
Taylor --> 3 parameter	**2.9**	**2.4**	**5.6**	**.20**	**1.00**
Taylor --> 2 parameter	**2.9**	**2.5**	**5.6**	**.21**	**1.02**
2 --> 3 parameter	**3.0**	**2.4**	**5.6**	**.20**	**1.00**

Notes: The syntax "*n*--> *m*" refers to the results from learning the transition from the *n*-parameter simple optimal rule to the *m*-parameter simple optimal rule. Losses are computed as discounted loss with a discount factor equal to 0.9875.

Now let us consider the decision by a strong policymaker, of whether to shift from a Taylor rule to the optimal 2-parameter rule as well as to the optimal 3-parameter rule. To separate the effects of optimality from complexity, we also consider moving from the optimal 2-parameter rule to the optimal 3-parameter rule. We conduct these exercises with full memory ($\lambda = 1$ in the second panel of Table 5), long memory (third panel), and short memory (bottom panel). The first thing to note about the results is that when agents have to learn about rule changes, short memory is a good thing. The loss for the short-memory cases are less than for the corresponding long-memory cases, which in turn are lower than the full-memory cases, regardless of which learning exercise one considers. Since the variability of $\hat{\beta}$ varies inversely with λ, this result is not trivial. It would stand to reason that greater variability of the perceived rule parameters in steady state would correspond with higher losses in the steady state. These higher steady-state losses would have to be netted off against the lower transitional losses as the steady state is approached. In fact, the difference in loss in the steady-state loss between the short-memory case and the full-memory case is negligible, meaning that, for these cases at least, only the transitional losses matter.

A second observation from Table 5 is that regardless of how slow the learning is, the strong policymaker is always willing to bear the transitional costs of moving from the Taylor rule to either the 3-parameter or 3-parameter optimal rules (compare the loss column of the first row with any row in the bottom three panels). More intriguingly, however, in some cases, the strong authority is better off moving to the optimal 2-parameter rule than moving directly to the 3-parameter rule. This is so even though the cost of moving from the optimal 2-parameter rule to the optimal 3-parameter rule is generally small. The reason is, of course, that the benefits are often smaller still.

Now let us examine the same experiment for weak preferences for inflation control, shown in Table 6. The mass of numbers in the table yield essentially one point: regardless of whether one begins from the Taylor rule or from the 2-parameter optimal (weak) rule, and regardless of the memory in the learning mechanism, there is very little difference in the loss. Simply put, neither the speed of learning, nor the complexity of the rule, makes a substantial difference in terms of welfare. To see why this is so, first consider the welfare loss to a weak policymaker from pursuing the Taylor rule. The loss is certainly consequential,[24] but given the vast distance the Taylor rule coefficients are from the optimal 2-parameter rule coefficients, the difference in performance seems small. Simply put, the loss function for the weak policymaker is very flat, or, to put the same observation another way, relatively crude control methods yield broadly similar economic performance, for weak preferences.

To some extent, this is a corollary of the point made in subsection 4.1 above with regard to Figure 1. There we noted that inflation control and interest-rate smoothing tend to be substitutes. There is no similar conflict for the weak policymaker. This can be seen in Figure 1 by comparing the very steep slopes of the trade-offs between the variability in the change in the federal funds rate and the variance of inflation in the neighborhood of the weak rules, with the much flatter slopes of the same curves in the neighborhood of the strong rules. Managing output fluctuations to the near-exclusion of other objectives is a relatively easy task. Because control performance does not depend in an important way on forward expectations, agents' prior expectations of what rule is in place is of second-order importance.

Taken together, the results shown in Table 5 and Table 6 send a cautionary message about how a monetary authority should select among candidate rules. For some preferences, the characteristics of rules matter a great deal for economic performance, as do the transition costs of moving from one rule to another. For other preferences, any vaguely sensible rule will perform reasonably well. Thus, no simple rule of thumb for monetary policy rule selection emerges.

24. The weak policymaker would be willing to permit the variance of output to rise by 0.3 in order to avoid using the Taylor rule. Recall from earlier that the strong policymaker would tolerate a 0.75 increase in inflation to avoid the same fate. (Inflation carries the same 0.8 weight in the strong policymaker's loss function as output does in the weak's loss function.)

5. Concluding Remarks

This paper has examined the implications for the design of monetary policy rules of the complexity of rules and the interaction of complexity and preferences with the process of learning by private agents of the inflation-targeting rule that is in place.

In particular, we took a small New Keynesian macroeconometric model and computed optimal simple rules for two sets of preferences: strong preferences for inflation control, where a substantial penalty is attached to inflation variability and only a small weight on output or instrument variability; and weak preferences for inflation control, where the same substantial weight is placed on output variability, and not on inflation or instrument control. Then we compared the stochastic performance of these policies that would have been optimal within a single regime, to two types of transition experiments. The first was the transition to more complex rules from simpler rules, within a single regime. The second was the transition between regimes for a simple optimal rule of given complexity.

Our four basic findings are: (1) learning should be expected to be a slow process. Even when agents 'forget' the past with extraordinary haste, it takes more than ten years for agents to learn the correct parameters of a new rule. (2) The costs of these perceptual errors can vary widely, depending on the rule that is initially in force, and on the preferences of the monetary authority. In particular, a strong inflation-targeting monetary authority tends to find high costs associated with the need for agents to learn a new (strong) rule that has been put in place. It follows that such an policymaker should be willing to take steps to identify his policy preferences to private agents. Paradoxically, a weak policymaker will sometimes benefit from being misperceived, posting a better economic performance than would have been the case if the optimal rule had been in place all along. This sharp contrast in results has to do with the multiplicity of sources of shocks to inflation, the nature of inflation in this model being a forward-looking variable, and the fact that inflation appears later in the chain of the monetary policy transmission mechanism than does output. (3) The performance, in steady state, of optimal two-parameter policy rules is not much worse than the performance of optimal three-parameter policy rules, at least for this model. Largely for this reason, some policymakers that would like to move from a suboptimal rule would be better off moving to the optimal 2-parameter rule, and forsaking the 3-parameter rule, than bearing the incremental costs of private agents having to learn the more complicated rule. (4) Faster learning is not necessarily better. When the monetary authority takes steps to 'actively teach' private agents that the rule has changed, agents' expectations converge more rapidly on the new and true parameters, but economic performance does not necessarily improve. This is because the policymaker himself must add instrument variability to the system in order to hasten the learning, and because initial benefits of more rapid learning are 'given back' when learning slows down later on.

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7. Appendix: Derivation of Optimal Rules

This appendix derives simple optimal rules and the optimal control rule, where the former turn out to be special cases of the latter.

A. State space representation of the model

In the main body of this paper, the following variable definitions are used: y_t is excess demand, π_t is goods inflation, rs_t is an interest rate—in the present case, the overnight borrowing rate under the control of the monetary authority, and c_t is contract inflation. The model is,

$$\begin{aligned} y_t &= \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 (rs_{t-1} - \pi_{t-1}) + u_{y,t}, & (1) \\ \pi_t &= \delta \pi_{t-1} + (1-\delta) c_t & (2) \\ c_t &= (1-\delta)(\pi_{t-1} + \gamma y_{t-1}) + \delta c_{t+1,t} + u_{\pi,t}, & (3) \end{aligned}$$

where $\phi_2 < 0$, $\phi_3 < 0$, and $c_{t+1,t}$ is the expected value of c_{t+1}, given information on period t. All variables are measured as deviations from equilibrium, implying that their steady states are zero. In addition to the above model, there is an equation representing the authority's policy rule

$$\begin{aligned} rs_t &= \pi_t + u_t, & (4) \\ &\equiv \pi_t - F x_t, \end{aligned}$$

where u_t is the control variable, F is a vector of constants to be determined, and x_t, is the state vector,

$$x_t = \begin{bmatrix} z_t \\ c_t \end{bmatrix},$$

where $z_t = [rs_{t-1}, y_{t-1}, \pi_{t-1}, y_{t-2}]'$, represents the four predetermined (inertial) variables in the system, and c_t is the forward-looking jump variable, where we make the rational expectations assumption that $c_{t+1,t}$ is consistent with the mathematical expectation of c_{t+1} obtained by solving the model. In matrix notation, the first-order autoregressive form of the equations in (1) -(4) is,

$$C_1 x_{t+1} = C_0 x_t + B_0 u_t + \epsilon_t, \qquad (5)$$

where $B_0 = [1, 0, 0, 0, 0]'$, $\epsilon_t = [0, u_{y,t}, 0, 0, u_{\pi,t}]'$, and C_1 and C_0 are 5×5 matrices,

$$C_1 = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -(1-\delta) \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\delta \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \phi_3 & \phi_1 & -\phi_3 & \phi_2 & 0 \\ 0 & 0 & \delta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & (1-\delta)\gamma & (1-\delta) & 0 & -1 \end{bmatrix}.$$

Premultiplying (5) by C_1^{-1}, yields the state transition equations,

$$x_{t+1} = Cx_t + Bu_t + \eta_t, \tag{6}$$

where

$$C = \begin{bmatrix} 0 & -\frac{\gamma(1-\delta)^2}{\delta} & \delta - \frac{(1-\delta)^2}{\delta} & 0 & \frac{1-\delta}{\delta} \\ \phi_3 & \phi_1 & -\phi_3 & \phi_2 & 0 \\ 0 & -\frac{\gamma(1-\delta)^2}{\delta} & \delta - \frac{(1-\delta)^2}{\delta} & 0 & \frac{1-\delta}{\delta} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{\gamma(1-\delta)}{\delta} & -\frac{1-\delta}{\delta} & 0 & \frac{1}{\delta} \end{bmatrix},$$

and $B = C_1^{-1}B_0 = B_0$, $\eta_t = C_1^{-1}\epsilon_t$, $E\eta_t\eta_t' = \Sigma_\eta = C_1^{-1}\Sigma_\epsilon(C_1^{-1})'$.

In the above representation, the state vector, x_{t+1}, evolves from its preceding value in t via the transition matrix, C, modified by the effect of the control, u_t, and the unforeseen demand and supply shocks, η_t.

The policy authority targets inflation, output, and changes in the short-term interest rate. Accordingly, define the vector, $s_t = [rs_t - rs_{t-1}, y_t, \pi_t]'$. Given the definition (4), s_t obeys the mapping,

$$s_t = Mx_t + M_u u_t, \tag{7}$$

where,

$$M = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad M_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The central bank seeks to minimize the expected present value of a weighted sum of squared deviations of inflation, output, and changes in the interest rate from their respective (zero) targets. Defining the diagonal 3×3 performance metric, Ψ_s,

$$\Psi_s = \begin{bmatrix} \psi_{\Delta rs} & 0 & 0 \\ 0 & \psi_y & 0 \\ 0 & 0 & \psi_\pi \end{bmatrix},$$

the expected loss to be minimized is,

$$\begin{aligned} EW_0 &= \frac{1}{2} E \sum_{t=0}^{\infty} \rho^t s_t' \Psi_s s_t, \\ &= \frac{1}{2(1-\rho)} tr(\Psi_s \Sigma_s), \end{aligned} \tag{8}$$

where E is the expectations operator, as before, $0 < \rho \leq 1$ is the discount factor, and Σ_s is the unconditional covariance matrix of s, so that, asymptotically, the authority is seeking to minimize a weighted sum of the unconditional variances of the three target variables.

In light of (7), the expected loss can be re-expressed as a function of the entire state vector, x_t, and the control variable, u_t,

$$EW_0 = \frac{1}{2} \sum_{t=0}^{\infty} \rho^t [x_t' \Psi x_t + 2x_t' U u_t + u_t' R u_t], \tag{9}$$

where

$$\begin{aligned} \Psi &= M' \Psi_s M \\ U &= M' \Psi_s M_u \\ R &= M_u' \Psi_s M_u. \end{aligned}$$

Standard optimal control packages assume no discounting, $\rho = 1$, and no crossproducts, $U = 0$. However, a simple transformation of the variables allows us to translate the problem with crossproducts and discounting into a conventional optimal control problem. To this end, define,

$$\begin{aligned} \hat{u}_t &= (1-\rho)^{1/2} \rho^{t/2} (u_t + R^{-1} U' x_t) \\ \hat{x}_t &= (1-\rho)^{1/2} \rho^{t/2} x_t, \end{aligned}$$

and observe that

$$\hat{u}_t' R \hat{u}_t \;=\; (1-\rho)\rho^t [x_t' u_t'] \begin{bmatrix} UR^{-1}U' & U \\ U' & R \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix},$$

so that

$$(1-\rho)\rho^t [x_t' u_t'] \begin{bmatrix} \Psi & U' \\ U & R \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} = \hat{x}_t \bar{\Psi} \hat{x}_t + \hat{u}_t' R \hat{u}_t,$$

where

$$\bar{\Psi} = \Psi - UR^{-1}U'.$$

Further, defining

$$\begin{aligned} \bar{C} &= \rho^{1/2}(C - BR^{-1}U') \\ \bar{B} &= \rho^{1/2}B, \end{aligned}$$

we may rewrite (9)

$$EW_0 \;=\; \frac{1}{2} E \sum_{t=0}^{\infty} [\hat{x}_t' \bar{\Psi} \hat{x}_t + \hat{u}_t' R \hat{u}_t], \tag{10}$$

subject to

$$\hat{x}_{t+1} = \bar{C}\hat{x}_t + \bar{B}\hat{u}_t + \hat{\eta}_t, \tag{11}$$

where $\hat{\eta}_t = (1-\rho)^{1/2} \rho^{\frac{t+1}{2}} \eta_t$.

B. Optimal control

In optimal control, we seek a vector, $\bar{F}$, satisfying

$$\hat{u}_t = -\bar{F}\hat{x}_t, \tag{12}$$

that minimizes the asymptotic expected loss (10) subject to (11). Substituting (12) for $\hat{u}_t$ in (10) and (11), EW_0 is, equivalently,

$$EW_0 = \frac{1}{2} tr[(\bar{\Psi} + \bar{F}' R \bar{F})\Sigma_x] + tr\{S[\Sigma_\eta - \Sigma_x + (\bar{C} - \bar{B}\bar{F})\Sigma_x(\bar{C} - \bar{B}\bar{F})']\} \tag{13}$$

where Σ_x is the asymtoptic covariance of x, [1]

$$\Sigma_x \;=\; \Sigma_\eta + (\bar{C} - \bar{B}\bar{F})\Sigma_x(\bar{C} - \bar{B}\bar{F})', \tag{14}$$

[1] To show that this is so, let $A = \bar{C} - \bar{B}\bar{F}$, so that (11) becomes

$$\hat{x}_{t+1} \;=\; A\hat{x}_t + \hat{\eta}_t.$$

and S is the 5×5 matrix of Lagrangian variables associated with the constraint (14). Differentiating (13) with respect to $\bar{F}$ and Σ_y, we determine the two equations familiar from the control literature,[2]

$$\begin{aligned} \bar{F} &= [R + \bar{B}'S\bar{B}]^{-1}\bar{B}'S\bar{C}, \\ S &= \bar{\Psi} + (\bar{C} - \bar{B}\bar{F})'S(\bar{C} - \bar{B}\bar{F}) + \bar{F}'R\bar{F}. \end{aligned}$$

Finally, a feedback law for the orginal state variables, x_t, is retrieved by observing that,

$$u_t = -(\bar{F} + R^{-1}U')x_t \equiv -Fx_t.$$

Formulation of an operational feedback rule is complicated by the fact that the optimal control rule, (12) as solved, contains the expectational variable, c_t, which itself, jumps with the selection of the rule. Based on a solution due to Backus and Driffill (1986), one can express the optimal policy as a function of solely the predetermined variables, z, by writing it first as a function of those predetermined variables and the costate variables, q, associated with the non-predetermined variable, c,

$$\begin{aligned} u_t &= -FH^{-1}\begin{bmatrix} z_t \\ q_t \end{bmatrix}, \\ &\equiv G_1 z_t + G_2 q_t, \end{aligned} \tag{15}$$

footnote 1 continued. Recursively substituting x into itself,

$$\hat{x}_{t+k} = A^k \hat{x}_t + (1-\rho)^{1/2} \sum_{i=1}^{k} \rho^{\frac{t+k-i}{2}} A^{k-i} \eta_{t+i}.$$

Thus, as k becomes large, the covariance of $\hat{x}_t$ is the covariance of x_t:

$$\Sigma_{\hat{x}} = \lim_{k \to \infty} (1-\rho) \sum_{i=0}^{k} \rho^i A^i \Sigma_\eta (A^i)' = \Sigma_x.$$

[2]Here we have exploited the facts that

$$\begin{aligned} \frac{\partial}{\partial B} tr(AB) &= \frac{\partial}{\partial B} tr(BA) = A' \\ \frac{\partial}{\partial B} tr(B'ABC) &= ABC + A'BC'. \end{aligned}$$

where

$$H = \begin{bmatrix} I & 0 \\ S_{21} & S_{22} \end{bmatrix},$$

and S_{21} and S_{22} are appropriately-dimensioned partitioned submatrices of S. The indices, '1' and '2' correspond to the predetermined and non-predetermined variables, respectively. Let $T = H(A - BF)H^{-1}$. Then the transition equation for z and q is,

$$\begin{bmatrix} z_{t+1} \\ q_{t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} z_t \\ q_t \end{bmatrix}.$$

Accordingly, q can be expressed as a difference equation driven by the predetermined variables, z_t,

$$q_{t+1} = T_{22}q_t + T_{21}z_t.$$

Given the policy rule (15), this can be written,

$$q_t = T_{21}z_{t-\tau} + T_{22}G_2^{-1}(u_t - G_1 z_t),$$

so that, solving for u_t, we obtain,

$$\begin{aligned} u_t &= G_2(G_2T_{22}^{-1})^{-}u_{t-1} + G_1z_t + G_2[T_{21} - (G_2T_{22}^{-1})^{-}G_1]z_{t-1}, \\ &\equiv \alpha_u u_{t-1} + \alpha_0 z_t + \alpha_1 z_{t-1}, \\ &\equiv Kw_t, \end{aligned} \tag{16}$$

where, w_t is the vector $w_t = [u_t, z_t, z_{t-1}]'$, and α_0 and α_1 are 1×4 vectors corresponding to the dimension of z_t.[3] Note that $(G_2T_{22}^{-1})^{-1}$ is a pseudo-inverse if the number of jumpers, m, exceeds the number of instruments, k, in the model, as is typically the case in macroeconomic models.[4] In the model of this paper, there is only one non-predetermined variable and one control variable, so $G_2T_{22}^{-1}$ is a scalar.

Finally, the optimal steady-state rule can be expressed as the seven-parameter rule,

$$\begin{aligned} rs_t - \pi_t &= \beta_{rs}(rs_{t-1} - \pi_{t-1}) + \beta_{rs_{t-2}}(rs_{t-2} - \pi_{t-2}) \\ &+ \beta_\pi \pi_{t-1} + \beta_{\pi_{t-2}}\pi_{t-2} \\ &+ \beta_y y_{t-1} + \beta_{y_{t-2}}y_{t-2} + \beta_{y_{t-3}}y_{t-3}, \end{aligned}$$

[3] Levine (1991) has pointed out that (16) is equivalent to an error-correction rule.

[4] A pseudo-inverse of a non-square matrix requires a singular-value decomposition of the matrix to be inverted and can be obtained, for example, with the MATLAB function *pinv.m*.

where

$$\begin{aligned}
\beta_{rs} &= \alpha_u + \alpha_{01}, \\
\beta_{rs_{t-2}} &= \alpha_{11}, \\
\beta_{\pi} &= \alpha_{01} + \alpha_{03}, \\
\beta_{\pi_{t-2}} &= \alpha_{11} + \alpha_{13}, \\
\beta_{y} &= \alpha_{02}, \\
\beta_{y_{t-2}} &= \alpha_{04} + \alpha_{12}, \\
\beta_{y_{t-3}} &= \alpha_{14}.
\end{aligned}$$

C. Simple Optimal Policy

The "simple" policy rules considered in this paper are versions of (16),

$$u_t = \tilde{D}w_t \equiv \tilde{D}Jx_t,$$

where J is a 7×6 matrix that maps x_t into w_t, and $\tilde{D}$ has the same dimension as D in (16) but may contain elements that are restricted to zero. Define the transfer function, $G(L)$, mapping the disturbances, η_t, onto the output vector, s_t: $G(L) = (M + M_u\tilde{D}J)[I - (A + B\tilde{D}J)L]^{-1}$, where L is the lag operator: $Lx_t = x_{t-1}$. Then, given a selection of k elements in $\tilde{D}$ that are allowed to change, an optimal $k-$parameter rule is determined by constrained optimization such that $\tilde{D}$ satisfies,

$$EW_0 = \min_{\tilde{D}} tr[G(1)'\Psi_s G(1)\Sigma_\eta],$$

subject to

$$s_{t+1} = G(L)\eta_t.$$

The minimum is determined iteratively, where, for this application, we used MATLAB's constrained optimization function, *constr.m*, where, with each $i-$th trial $\tilde{D}^i$, the model is solved backward, using the Anderson and Moore (1985) generalized saddlepath procedure, until a minimum is determined.

D. The Rules Compared

The following table compares the optimal control rule with two versions of simple optimal policy. As in the text, we contrast "weak" inflation preferences, $\psi_{\Delta r} = .1, \psi_y = .8$, and $\psi_\pi = .1$, that favor output stabilization, and "strong" inflation preferences, $\psi_{\Delta r} = .1, \psi_y = .1$, and $\psi_\pi = .8$, that place relatively more weight on inflation stabilization.

Simple Optimal Rules and the Optimal Control Rule

	WEAK INFLATION TARGETING			STRONG INFLATION TARGETING		
	Simple Rules		Optimal Control	Simple Rules		Optimal Control
	2-parameter	3-parameter		2-parameter	3-parameter	
$rs_{t-1} - \pi_{t-1}$		$-.33$	.97		.27	1.11
y_{t-1}	1.18	1.55	1.79	.73	.60	1.04
π_{t-1}	.40	.53	.26	1.70	1.35	.96
y_{t-2}			-2.28			-1.18
c_t						
$rs_{t-2} - \pi_{t-2}$			$-.10$			$-.27$
π_{t-2}			$-.16$			$-.58$
y_{t-3}			.64			.31
σ_y	2.5	2.4	2.4	3.0	3.1	3.1
σ_π	3.0	3.1	2.7	2.1	2.1	2.0
σ_r	5.7	5.7	5.5	6.4	6.5	6.4
Loss	6.7	6.6	6.4	5.6	5.4	5.3