ONLINE SUPPLEMENT
Spectral backtests unbounded and folded*

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1 Standardizing the Fernández-Steel t distribution

Recall the density of the Fernández-Steel t (FS-t) distribution:

$$f_{FS}(x;\gamma,\zeta) = \frac{2\gamma}{\sigma(1+\gamma^2)} \begin{cases} f_t\left(\frac{\gamma(x-\mu)}{\sigma};\zeta\right) & x \leqslant \mu \\ f_t\left(\frac{x-\mu}{\gamma\sigma};\zeta\right) & x > \mu \end{cases} \tag{OS.1}$$

for location parameter μ and scale parameter σ and where f_t is the density of the t distribution. In this appendix, we show how these parameters are specified to obtained a standardized distribution of mean zero and unit variable.

Let

$$\begin{aligned} m_1(\zeta) &= \frac{2\Gamma(\frac{\zeta+1}{2})\zeta}{\sqrt{\pi\zeta}\Gamma(\frac{\zeta}{2})(\zeta-1)} \\ m_2(\zeta) &= \frac{\zeta}{\zeta-2}. \end{aligned}$$

These are the first and (non-central) second moments of the absolute value of a Student t distributed random variable with ζ degrees of freedom. In the case of a normal distribution

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they simplify to $m_1(\infty) = \sqrt{2/\pi}$ and $m_2(\infty) = 1$. The required scaling and location parameters in (OS.1) to give a distribution with mean 0 and variance 1 are

$$\begin{aligned}\sigma(\zeta,\gamma) &= \left((m_2(\zeta) - m_1(\zeta)^2)\left(\gamma^2 + \gamma^{-2}\right) + 2m_1(\zeta)^2 - m_2(\zeta)\right)^{-1/2} \\ \mu(\zeta,\gamma) &= -\sigma(\zeta,\gamma)m_1(\zeta)\left(\gamma - \gamma^{-1}\right).\end{aligned}$$

2 Special cases of the hypergeometric ${}_3F_2(1)$ function

We seek solutions to the second moments and cross-moments of the transformed PIT values when the standardized kernel densities take the form

$$\tilde{g}_\nu(u) = u^{a-1}(1-u)^{b-1}$$

for parameters $(a > 0, b > -1/2)$. As discussed in Appendix C of the main text, this involves the integral

$$M(a_1, b_1, a_2, b_2) = \int_0^1 (1-u)\tilde{g}_1(u)\tilde{G}_2(u)du \tag{OS.2}$$

where $\tilde{G}_i$ denotes the transform function for kernel i and has parameters (a_i, b_i). The general solution to M can be written in two ways:

$$M(a_1, b_1, a_2, b_2) = \frac{B(a_1 + a_2, 1 + b_1)}{a_2}{}_3F_2(a_2, a_1 + a_2, 1 - b_2; 1 + a_2, 1 + a_1 + a_2 + b_1; 1) \tag{OS.3}$$

$$= \frac{B(a_1 + a_2, 1 + b_1 + b_2)}{a_2}{}_3F_2(1, a_1 + a_2, a_2 + b_2; 1 + a_2, 1 + a_1 + a_2 + b_1 + b_2; 1) \tag{OS.4}$$

In this appendix we document special cases for which the ${}_3F_2(1)$ terms have known closed-form solution.

2.1 Case $b_1 = 0$

When $b_1 = 0$ and $b_2 \neq 0$, we apply Wolfram Research (2023, 07.27.03.0119.01) to (C.3) to obtain

$$M(a_1, 0, a_2, b_2) = \frac{1}{a_1}\left(B(a_2, b_2) - B(a_1 + a_2, b_2)\right) \tag{OS.5}$$

When $b_1 = b_2 = 0$, we instead apply Wolfram Research (2023, 07.27.03.0035.01) to (OS.4):

$$M(a_1, 0, a_2, 0) = \frac{1}{a_1}\left(\psi(a_1 + a_2) - \psi(a_2)\right) \tag{OS.6}$$

where ψ is the digamma function. One can derive this by noting that Wolfram Research (2023, 06.14.26.0001.01 and 06.18.06.0004.01) together imply the limit

$$\lim_{b\to 0}\left(\frac{1}{b} - B(a,b)\right) = \psi(a) + \gamma$$

where γ is the Euler-Mascheroni constant.

2.2 Case $b_1 = n \in \mathbb{N}$

Binomial expansion of the term $(1-u)^{b_1}$ in the integral form (C.5) of M leads to

$$M(a_1, n, a_2, b_2) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} M(a_1 + k, 0, a_2, b_2) \tag{OS.7}$$

This approach would be practical for small values of n.

2.3 Case $a_2 = n \in \mathbb{N}$

For the moment, assume $b_2 \neq 0$. We have from Wolfram Research (2023, 06.19.03.0003.01):

$$B(u; n, b) = B(n, b) \left(1 - (1-u)^b \sum_{k=0}^{n-1} \frac{(b)_k u^k}{k!} \right) \tag{OS.8}$$

where $(b)_k$ is Pochhammer's notation defined recursively by $(b)_0 = 1$, $(b)_{m+1} = (b+m) \cdot (b)_m$. This leads to

$$M(a_1, b_1, n, b_2) = \frac{(n-1)!}{(b_2)_n} \left(B(a_1, 1 + b_1) - \sum_{k=0}^{n-1} \frac{(b_2)_k}{k!} B(a_1 + k, 1 + b_1 + b_2) \right). \tag{OS.9}$$

This can be computed more effectively in recursive form:

$$M(a_1, b_1, 1, b_2) = \frac{1}{b_2} \left(B(a_1, 1 + b_1) - B(a_1, 1 + b_1 + b_2) \right) \tag{OS.10}$$

$$M(a_1, b_1, m+1, b_2) = \frac{1}{b_2 + m} \left(m \, M(a_1, b_1, m, b_2) - B(a_1 + m, 1 + b_1 + b_2) \right) \tag{OS.11}$$

To take the limit of (OS.10) as $b_2 \to 0$, we have by Wolfram Research (2023, 06.18.06.0013.01) that

$$B(a, b+z) = B(a, b)(1 + z(\psi(b+1) - \psi(a+b+1)) + \mathcal{O}(z^2))$$

which implies

$$M(a_1, b_1, 1, 0) = B(a_1, 1 + b_1) \left(\psi(1 + a_1 + b_1) - \psi(1 + b_1) \right) \tag{OS.12}$$

Observe that (OS.11) is well-behaved at $b_2 = 0$.

2.4 Case $b_2 = n \in \mathbb{N}$

We have from Wolfram Research (2023, 06.19.03.0001.01):

$$B(u; a, 1+m) = B(a, 1+m) \, u^a \sum_{k=0}^{m} \frac{(a)_k (1-u)^k}{k!} \tag{OS.13}$$

Substituting into the integral for M:

$$M(a_1, b_1, a_2, n) = \frac{(n-1)!}{(a_2)_n} \sum_{k=0}^{n-1} \frac{(a_2)_k}{k!} B(a_1 + a_2, b_1 + k + 1). \tag{OS.14}$$

This could also be expressed compactly in recursive form. An especially simple subcase is

$$M(a_1, b_1, a_2, 1) = \frac{1}{a_2} B(a_1 + a_2, b_1 + 1). \tag{OS.15}$$

3 Examples of TLSF score tests

Test based on normal distribution: We have $R = \Phi$, $\rho = \phi$, $\lambda_\rho(x) = x$, and $\lambda'_\rho(x) = 1$. Consequently, integrals of type $\mathcal{B}_{\rho,k}$ have solution $\mathcal{B}_{\rho,0}(\alpha) = \alpha$, $\mathcal{B}_{\rho,1}(\alpha) = \mathcal{A}_{\rho,0}(\alpha)$, and $\mathcal{B}_{\rho,2}(\alpha) = \mathcal{A}_{\rho,1}(\alpha)$.

Test based on logistic distribution: When $R(x)$ is the logistic function $S(x)$, we have $\rho(x) = S(x)S(-x)$, $\lambda_\rho(x) = S(x) - S(-x)$ and $\lambda'_\rho(x) = 2\rho(x)$. The inverse cdf is $R^{-1}(p) = \ln(p/(1-p))$, which leads to the convenient expressions $\rho\left(R^{-1}(\alpha)\right) = \alpha(1-\alpha)$, $C_1\left(R^{-1}(\alpha)\right) = 1-\alpha$, $\bar{C}_1\left(R^{-1}(\alpha)\right) = -\alpha$, $C_2\left(R^{-1}(\alpha)\right) = \bar{C}_2\left(R^{-1}(\alpha)\right) = -\alpha(1-\alpha)$.

Integrals of type $\mathcal{B}_{\rho,k}$ have solutions

$$\mathcal{B}_{\rho,0}(\alpha) = \frac{1}{3}\alpha^2(3 - 2\alpha) \tag{OS.16}$$

$$\mathcal{B}_{\rho,1}(\alpha) = \begin{cases} \frac{1}{3}\left(\alpha(1-\alpha) + \ln(1-\alpha)\right) + \mathcal{B}_{\rho,0}(\alpha)R^{-1}(\alpha) & \alpha < 1 \\ 0 & \alpha = 1 \end{cases} \tag{OS.17}$$

$$\mathcal{B}_{\rho,2}(\alpha) = \begin{cases} \frac{-2}{3}\left(\alpha + \mathrm{Li}_2\left(\frac{-\alpha}{1-\alpha}\right)\right) + 2\mathcal{B}_{\rho,1}(\alpha)R^{-1}(\alpha) - \mathcal{B}_{\rho,0}(\alpha)R^{-1}(\alpha)^2 & \alpha < 1 \\ \frac{2}{3}\left(\frac{\pi^2}{6} - 1\right) & \alpha = 1. \end{cases} \tag{OS.18}$$

The dilogarithm function Li_2 is available in the GSL package.

For the lower bound on permissible α_1, (14) simplifies to the condition $xR(x) = 1$, which has solution $x_1 = W_0(1/e) + 1$, where W_0 here denotes the Lambert W-function. This leads to the bound $\alpha_1 \geqslant 1/(1 + W_0(1/e)) \approx 0.782$.

Tests based on Gumbel distribution: For the standard Gumbel, we have $R(x) = \exp(-\exp(-x))$ and $\rho(x) = \exp(-x)R(x)$, so $\lambda_\rho(x) = 1 - \exp(-x) = 1 - C_1(x)$. The inverse cdf is $R^{-1}(p) = -\ln(-\ln(p))$, which leads to $\rho\left(R^{-1}(\alpha)\right) = -\alpha\ln(\alpha)$ and $\lambda_\rho\left(R^{-1}(\alpha)\right) = 1 + \ln(\alpha)$. For $\mathcal{B}_{\rho,0}(\alpha)$ and $\mathcal{B}_{\rho,1}(\alpha)$ we have

$$\mathcal{B}_{\rho,0}(\alpha) = \begin{cases} 0 & \alpha = 0 \\ \alpha(1 - \ln(\alpha)) & \alpha > 0 \end{cases}$$

$$\mathcal{B}_{\rho,1}(\alpha) = \begin{cases} \mathrm{li}(\alpha) - \alpha + \mathcal{B}_{\rho,0}(\alpha)R^{-1}(\alpha) & \alpha < 1 \\ \gamma - 1 & \alpha = 1 \end{cases}$$

where li is the logarithmic integral function and γ is the Euler-Mascheroni constant. $\mathcal{B}_{\rho,2}(\alpha)$ is best solved numerically.

For the complementary Gumbel, we apply Remark (3.6) to obtain R_c, λ_c, etc., and equations (D.16)–(D.17) to provide solutions to $\mathcal{A}^c_{\rho,k}$ and $\mathcal{B}^c_{\rho,k}$.

Tests based on logistic-beta distribution: The logistic-beta distribution has density

$$\rho(x; a, b) = \frac{S(x)^a S(-x)^b}{B(a, b)}$$

for parameters $a > 0, b > 0$. $S(x) = 1/(1 + \exp(-x))$ is the logistic function and $B(a, b)$ is the beta function.

The logistic function satisfies $S(-x) = 1 - S(x)$ and $S'(x) = S(x)S(-x)$ from which it follows that

$$\begin{aligned} \lambda_\rho(x) &= bS(x) - aS(-x) \\ \lambda'_\rho(x) &= (a + b)S(x)S(-x) > 0 \end{aligned}$$

The cumulants of the distribution are

$$\kappa_m(a, b) = \psi^{(m-1)}(a) + (-1)^m \psi^{(m-1)}(b)$$

where $\psi^{(j)}$ is the j^{th} derivative of the digamma function. An implication is that the skew of the distribution has the same sign as $a - b$, since $\psi^{(2)}(x)$ is strictly increasing for $x > 0$.

The cdf is a composition of the beta cdf $\mathcal{I}(z; a, b)$ and the logistic function, i.e., $R(x) = \mathcal{I}(S(x); a, b)$. The inverse cdf is therefore

$$R^{-1}(p) = \text{logit}(\mathcal{I}^{-1}(p; a, b))$$

where $\text{logit}(u) = \ln(u/(1 - u))$. The well-known symmetry for the beta distribution implies a symmetry of the same form for the inverse cdf:

$$1 - \mathcal{I}^{-1}(p; a, b) = \mathcal{I}^{-1}(1 - p; b, a)$$

Consequently, we can write

$$R^{-1}(p) = \ln\left(\mathcal{I}^{-1}(p; a, b)\right) - \ln\left(\mathcal{I}^{-1}(1 - p; b, a)\right). \tag{OS.19}$$

Observe that

$$\begin{aligned} \lambda'_\rho(x; a, b)\rho(x; a, b) &= \frac{(a + b)}{B(a, b)} S(x)^{a+1} S(-x)^{b+1} \\ &= \frac{(a + b)B(a + 1, b + 1)}{B(a, b)} \rho(x; a + 1, b + 1) = \frac{ab}{1 + a + b} \rho(x; a + 1, b + 1) \end{aligned}$$

which implies that

$$\begin{aligned} \mathcal{B}_{\rho,0}(1) &= \frac{ab}{1 + a + b} \\ \mathcal{B}_{\rho,1}(1) &= \mathcal{B}_{\rho,0}(1)\kappa_1(a + 1, b + 1) \\ \mathcal{B}_{\rho,2}(1) &= \mathcal{B}_{\rho,0}(1)\left(\kappa_2(a + 1, b + 1) + \kappa_1(a + 1, b + 1)^2\right) \end{aligned}$$

For any α, we have

$$\mathcal{B}_{\rho,0}(\alpha) = \mathcal{B}_{\rho,0}(1) R\left(R^{-1}(\alpha; a, b); a+1, b+1\right).$$

For $k = 1, 2$, we can obtain solutions to $\mathcal{B}_{\rho,k}(\alpha)$ in terms of ${}_3F_2$ and ${}_4F_3$ hypergeometric functions. In practice, it is easiest to pre-calculate interpolation tables for these functions for select values of (a, b). The numeric integrals appear to be straightforward and knowing the end-value of $\mathcal{B}_{\rho,k}(1)$ ensures a high degree of accuracy.

One special case of possible special interest is the skew sech distribution. The density is usually written

$$\rho^{ss}(x; \theta) = \frac{\cos(\theta)}{2} \exp(\theta x) \operatorname{sech}\left(\frac{\pi}{2} x\right)$$

where the skew parameter $|\theta| < \pi/2$ and $\theta = 0$ corresponds to the standard sech distribution. We can easily show that if X is distributed skew sech with parameter θ then πX is distributed logistic-beta with parameters $(a, 1-a)$ for $a = 1/2 + \theta/\pi \in (0, 1)$.

References

Wolfram Research, "Mathematical Functions Site," https://functions.wolfram.com/, as of 2023-07-23 2023.